Flexible list colorings: Maximizing the number of requests satisfied

Hemanshu Kaul

Illinois Institute of Technology

www.math.iit.edu/∼kaul

kaul@iit.edu

Joint work with
Rogers Mathew (Indian Inst of Tech Hyderabad)
Jeffrey Mudrock (U of South Alabama)
Michael Pelsmajer (Illinois Tech)

Precoloring Extension Problem

- Recall proper k-coloring, proper k-list coloring, $\chi(G)$, $\chi_\ell(G)$.

Precoloring Extension Problem

- Is there a proper coloring of a graph G subject to some of the vertices having prescribed colors? A function r with non-empty domain $D \subseteq V(G)$ and co-domain of a palette of colors.

Precoloring Extension Problem

- Is there a proper coloring of a graph G subject to some of the vertices having prescribed colors? A function r with non-empty domain $D \subseteq V(G)$ and co-domain of a palette of colors.
- This is a classical problem that has been studied under many contexts. Its not always possible to have such a proper coloring, so typically we look for restrictions on the structure induced by the precolored vertices, etc.

Precoloring Extension Problem

- Is there a proper coloring of a graph G subject to some of the vertices having prescribed colors? A function r with non-empty domain $D \subseteq V(G)$ and co-domain of a palette of colors.

- What if such a precoloring does not extend, can we instead ask for a coloring which matches the precoloring on many vertices (say, on a constant fraction of the precolored vertices)?

Precoloring Extension Problem

- Is there a proper coloring of a graph G subject to some of the vertices having prescribed colors? A function r with non-empty domain $D \subseteq V(G)$ and co-domain of a palette of colors.

- What if such a precoloring does not extend, can we instead ask for a coloring which matches the precoloring on many vertices (say, on a constant fraction of the precolored vertices)?

- Always possible, by permuting the colors in a k-coloring of G, we can easily obtain a k-coloring of G that matches r on at least $|dom(r)|/k$ vertices.

List Coloring with Requests

- Dvořák, Norin, and Postle (2019): A proper list coloring, but a preferred color is given for some subset of vertices and we wish to color as many vertices in this subset with its preferred colored as possible; a flexible version of the classical precoloring extension problem.

List Coloring with Requests

- Given a graph G and a list assignment L of G.
  A request of L is a function r with non-empty domain $D \subseteq V(G)$ such that $r(v) \in L(v)$ for each $v \in D$.
  For any $\epsilon \in (0, 1]$, (G, L, r) is ϵ-satisfiable if there exists a proper L-coloring f of G such that $f(v) = r(v)$ for at least $\epsilon|D|$ vertices in D.

List Coloring with Requests

- Given a graph G and a list assignment L of G.
  A request of L is a function r with non-empty domain $D \subseteq V(G)$ such that $r(v) \in L(v)$ for each $v \in D$.
  For any $\epsilon \in (0, 1]$, (G, L, r) is ϵ-satisfiable if there exists a proper L-coloring f of G such that $f(v) = r(v)$ for at least $\epsilon|D|$ vertices in D.

- (G, L) is ϵ-flexible if (G, L, r) is ϵ-satisfiable whenever r is a request of L.
  G is (k, ϵ)-flexible if (G, L) is ϵ-flexible whenever L is a k-assignment for G.

List Coloring with Requests

- (G, L) is ϵ-flexible if (G, L, r) is ϵ-satisfiable whenever r is a request of L.
  G is (k, ϵ)-flexible if (G, L) is ϵ-flexible whenever L is a k-assignment for G.

- If G is (k, ϵ)-flexible, then it immediately follows:
  (i) G is (k', ϵ')-flexible for any $k' \geq k$ and $\epsilon' \leq \epsilon$;
  (ii) any spanning subgraph H of G is (k, ϵ)-flexible;
  (iii) G is k-choosable.

Previous works

- Dvořák, Norin, and Postle mostly focused on d-degenerate graphs, which are known to be $(d+1)$-choosable. They showed that d-degenerate graphs are all $(d+2, \epsilon)$-flexible for some $\epsilon > 0$.
- The main open problem Dvořák-Norin-Postle asked was: Are all d-degenerate graphs, $(d+1, \epsilon(d))$-flexible for some $\epsilon(d) > 0$?

Previous works

- The main open problem Dvořák-Norin-Postle asked was: Are all d-degenerate graphs, $(d+1, \epsilon(d))$-flexible for some $\epsilon(d) > 0$?

- They showed there exists an $\epsilon > 0$ such that every planar graph G is $(6, \epsilon)$-flexible.

- Flexible list coloring has been studied extensively for planar graphs that are 5-choosable, and for restricted subclasses of planar graphs that are k-choosable with $k < 5$. Several papers on (k, ϵ)-flexibility, with $k \in \{5, 4, 3\}$, of planar graphs with large enough girth or excluding certain cycles.

Motivation for our work

- Find the largest possible ϵ for which G is (k, ϵ)-flexible; that is, one would prefer to have a larger portion of the requested colors on vertices satisfied.

Motivation for our work

- Find the largest possible ϵ for which G is (k, ϵ)-flexible; that is, one would prefer to have a larger portion of the requested colors on vertices satisfied.

- Only previous result of this flavor is the following.

  Bradshaw, T. Masařík, L. Stacho (2022): Let G be a connected graph with $\Delta(G) \geq 3$ that is not a copy of $K_{\Delta(G)+1}$. Then, G is $(\Delta(G), 1/(6\Delta(G)))$-flexible. Moreover, $1/(6\Delta(G))$ is within a constant factor of being best possible.

Improving older results - I

Theorem (K., Mathew, Mudrock, Pelsmajer (2022+))

Suppose G is d-degenerate. Then, G is $(d+2, \frac{1}{2^{d+1}})$-flexible.

Improving older results - I

Theorem (K., Mathew, Mudrock, Pelsmajer (2022+))
Suppose G is d-degenerate. Then, G is $(d+2, \frac{1}{2^{d+1}})$-flexible.

- Proved using a randomized algorithm:
  We can order vertices of G as $v_1, \ldots, v_n$ such that for all $i \in [n]$, v_i has at most d neighbors v_j with $j > i$, and order each list of colors, $L(v_i)$, such that the requested color (if it exists) is the first color. Now pick uniformly from the first two available colors as we color the vertices in order.

Improving older results - I

Theorem (K., Mathew, Mudrock, Pelsmajer (2022+))

Suppose G is d-degenerate. Then, G is $(d+2, \frac{1}{2^{d+1}})$*-flexible.*

- This improves the following:

Theorem (Dvořák, Norin, and Postle (2019))

Suppose G is d-degenerate. Then,
G is $\left(d+2, \frac{1}{(d+2)^{(d+1)^2}}\right)$*-flexible.*

Improving older results - I

Theorem (K., Mathew, Mudrock, Pelsmajer (2022+))

Suppose G is d-degenerate. Then, G is $(d+2, \frac{1}{2^{d+1}})$-flexible.

Theorem (K., Mathew, Mudrock, Pelsmajer (2022+))

Let G be an s-choosable graph. Then, G is $(s+1, 1/\chi(G^2))$-flexible.

Improving older results - I

Theorem (K., Mathew, Mudrock, Pelsmajer (2022+))
Suppose G is d-degenerate. Then, G is $(d+2, \frac{1}{2^{d+1}})$-flexible.

Theorem (K., Mathew, Mudrock, Pelsmajer (2022+))
Let G be an s-choosable graph. Then, G is $(s+1, 1/\chi(G^2))$-flexible.

This yields an improvement on our first theorem above for d-degenerate graphs with maximum degree $\Delta < 2^d/d$.

Corollary (K., Mathew, Mudrock, Pelsmajer (2022+))
Let G be a d-degenerate graph with maximum degree Δ. If G is s-choosable, then G is $(s+1, 1/(\Delta(2d-1)+d-d^2+1))$-flexible.

Improving older results - II

- If we focus solely on *k* and allow arbitrarily small $\epsilon > 0$, then our colorings need only satisfy the color request at a single vertex. Then, without loss of generality, we need to only study requests with domain of size 1, as those have the most restrictive requirement.

Improving older results - II

- If we focus solely on k and allow arbitrarily small $\epsilon > 0$, then our colorings need only satisfy the color request at a single vertex. Then, without loss of generality, we need to only study requests with domain of size 1, as those have the most restrictive requirement.

- Dvořák, Norin, and Postle say "*A necessary condition for flexibility is that requests with singleton domain can be satisfied. Coming back to the case of d-degenerate graphs with lists of size $d + 1$, even proving this necessary condition is non-trivial and we can only do it in the special case that $d + 1$ is a prime.*"

Theorem (Dvořák, Norin, and Postle (2019))

Let $d \geq 2$ such that $\underline{d + 1 \text{ is a prime}}$. If G is a d-degenerate graph, L is a $(d + 1)$-assignment, and r is a request for G with domain of size 1, then (G, L, r) is 1-satisfiable.

Improving older results - II

Theorem (Dvořák, Norin, and Postle (2019))

Let $d \geq 2$ such that $\underline{d+1 \text{ is a prime}}$. If G is a d-degenerate graph, L is a $(d+1)$-assignment, and r is a request for G with domain of size 1, then (G, L, r) is 1-satisfiable.

Using the Alon-Tarsi Theorem we are able to extend their Theorem to all *d* for bipartite *d*-degenerate graphs.

Theorem (K., Mathew, Mudrock, Pelsmajer (2022+))

For any <u>bipartite</u> d-degenerate graph G with a $(d+1)$-list assignment L and request r with domain D of size 1, (G, L, r) is 1-satisfiable.

Improving older results - II

Theorem (K., Mathew, Mudrock, Pelsmajer (2022+))

For any bipartite d-degenerate graph G with a $(d+1)$-list assignment L and request r with domain D of size 1, (G, L, r) is 1-satisfiable.

We can order vertices of G as $v_1, \ldots, v_n$ such that for all $i \in [n]$, v_i has at most d neighbors v_j with $j > i$. Suppose that v_k is the vertex in the domain of r.
We show that there is a way, using Menger's Theorem, to orient the edges of G to obtain a digraph in which every vertex has out-degree at most d and v_k has out-degree zero. The result then follows from Alon-Tarsi.

Maximizing the number of vertex requests satisfied

For each graph G, what is the largest ϵ so that G is (k, ϵ)-flexible for some k?

Maximizing the number of vertex requests satisfied

- It is possible that $r(v)$ is the same color for all $v \in D$; for example, let L be the k-list assignment such that $L(v) = [k]$ for all $v \in V(G)$ and let $r(v) = 1$ for all $v \in D$. Then at most $\alpha(G[D])$ vertices in D will have their request fulfilled. So, $\epsilon \leq \min_{\emptyset \neq D \subseteq V(G)} \alpha(G[D])/|D|$ for any (k, ϵ)-flexible graph G.

Maximizing the number of vertex requests satisfied

- It is possible that $r(v)$ is the same color for all $v \in D$; for example, let L be the k-list assignment such that $L(v) = [k]$ for all $v \in V(G)$ and let $r(v) = 1$ for all $v \in D$. Then at most $\alpha(G[D])$ vertices in D will have their request fulfilled. So, $\epsilon \leq \min_{\emptyset \neq D \subseteq V(G)} \alpha(G[D])/|D|$ for any (k, ϵ)-flexible graph G.

- The Hall ratio of a graph G is $\rho(G) = \max_{\emptyset \neq H \subseteq G} \frac{|V(H)|}{\alpha(H)}$.

  The Hall ratio was first studied in 1990 by Hilton and Johnson Jr. under the name Hall-condition number in the context of list coloring. In the past 30 years, the Hall ratio has received much attention due to its connection with both list and fractional coloring.

Maximizing the number of vertex requests satisfied

For each graph G, what is the largest ϵ so that G is (k, ϵ)-flexible for some k?

The Hall ratio of a graph G is $\rho(G) = \max_{\emptyset \neq H \subseteq G} \frac{|V(H)|}{\alpha(H)}$.

Proposition (K., Mathew, Mudrock, Pelsmajer (2022+))

There exists k such that G is (k, ϵ)-flexible if and only if $\epsilon \leq 1/\rho(G)$.

Maximizing the number of vertex requests satisfied

For each graph G, what is the largest ϵ so that G is (k, ϵ)-flexible for some k?

The Hall ratio of a graph G is $\rho(G) = \max_{\emptyset \neq H \subseteq G} \frac{|V(H)|}{\alpha(H)}$.

Proposition (K., Mathew, Mudrock, Pelsmajer (2022+))

There exists k such that G is (k, ϵ)-flexible if and only if $\epsilon \leq 1/\rho(G)$.

- We define the list flexibility number of G, denoted $\chi_{flex}(G)$, to be the smallest k such that G is $(k, 1/\rho(G))$-flexible.

Maximizing the number of vertex requests satisfied

- We define the list flexibility number of G, denoted $\chi_{flex}(G)$, to be the smallest k such that G is $(k, 1/\rho(G))$-flexible.

List flexibility number

- $\chi_{flex}(G)$ is the smallest k such that G is $(k, 1/\rho(G))$-flexible.
- Is maximizing ϵ meaningfully different from a flexible list coloring with smaller $\epsilon > 0$? In other words, are there any graphs G and $k \in \mathbb{N}$ such that G is (k, ϵ)-flexible for some $\epsilon > 0$, but G is not $(k, 1/\rho(G))$-flexible? The following result shows that the answer is yes.

Proposition (K., Mathew, Mudrock, Pelsmajer (2022+))

Suppose $G = K_{3,7}$. Then, G is $(3, 1/10)$-flexible and $\chi_{flex}(G) > 3$.

List flexibility number

- $\chi_{flex}(G)$ is the smallest k such that G is $(k, 1/\rho(G))$-flexible.
- $\chi_\ell(G) \leq \chi_{flex}(G) \leq \Delta(G) + 1$.

- It follows that $\chi_{flex}(K_n) = n$ and $\chi_{flex}(C_k) = 3$ for odd k. It is natural to ask whether a Brooks-type theorem is true for χ_{flex} as well.
  Question: What are all the graphs G such that $\chi_{flex}(G) = \Delta(G) + 1$?

χ_{flex} vs degeneracy

- Can the Dvořák-Norin-Postle-Conjecture that d-degenerate graphs G are $(d+1, \epsilon(d))$-flexible for some $\epsilon(d) > 0$ be strengthened to $\chi_{flex}(G) \leq d+1$?
  Question: Does there exist a graph G with degeneracy d satisfying $\chi_{flex}(G) > d+1$?

χ_{flex} vs degeneracy

- Can the Dvořák-Norin-Postle-Conjecture that d-degenerate graphs G are $(d+1, \epsilon(d))$-flexible for some $\epsilon(d) > 0$ be strengthened to $\chi_{flex}(G) \leq d + 1$?
  Question: Does there exist a graph G with degeneracy d satisfying $\chi_{flex}(G) > d + 1$?

- Alon (2000) showed for any graph G with degeneracy d, $(1/2 - o(1)) \log_2(d + 1) \leq \chi_\ell(G)$ and is sharp up to a factor of 2. How sharp is this lower bound for $\chi_{flex}(G)$?
  Question: Suppose
  $\mathcal{F}(d) = \min\{\chi_{flex}(G) : \text{the degeneracy of } G \text{ is at least } d\}$.
  What is the asymptotic behavior of $\mathcal{F}(d)$ as $d \to \infty$?

χ_{flex} vs degeneracy

- Alon (2000) showed for any graph G with degeneracy d, $(1/2 - o(1))\log_2(d+1) \leq \chi_\ell(G)$ and is sharp up to a factor of 2. How sharp is this lower bound for $\chi_{flex}(G)$?
  Question: Suppose
  $\mathcal{F}(d) = \min\{\chi_{flex}(G) :$ the degeneracy of G is at least $d\}$.
  What is the asymptotic behavior of $\mathcal{F}(d)$ as $d \to \infty$?

- We are able to show $\mathcal{F}(d) = O(d)$ while Alon's result implies $\mathcal{F}(d) = \Omega(\log_2(d))$, as $d \to \infty$.

χ_{flex} vs degeneracy

Theorem (K., Mathew, Mudrock, Pelsmajer (2022+))

Suppose G is an n-vertex, s-choosable graph with $\chi_{flex}(G) = m$ *and* $n \geq 2$. *Let* $l = \lceil n/\rho(G) \rceil$. *Let* $J = G \vee G$. *Then, for any real number* $r > 2$,
$\chi_{flex}(J) \leq \max\left\{ \left\lceil l + \frac{\log_2(2n-l)}{1-H(1/r)} \right\rceil, \lceil r(s-1)+l \rceil, m \right\}.$

- H is the binary Entropy function.

χ_{flex} vs degeneracy

Theorem (K., Mathew, Mudrock, Pelsmajer (2022+))

Suppose G is an n-vertex, s-choosable graph with $\chi_{flex}(G) = m$ and $n \geq 2$. Let $l = \lceil n/\rho(G) \rceil$. Let $J = G \vee G$. Then, for any real number $r > 2$,
$\chi_{flex}(J) \leq \max \left\{ \left\lceil l + \frac{\log_2(2n-l)}{1-H(1/r)} \right\rceil, \lceil r(s-1) + l \rceil, m \right\}.$

Theorem (K., Mathew, Mudrock, Pelsmajer (2022+))

Suppose n is a positive even integer, and $G = P_n \vee P_n$. If $p \in (0,1)$ and $k > \max\{2, n/2\}$ satisfy $(n/2)(1-p)^{k-2} + np^{k-1-n/2}(p + (k-n/2)(1-p)) \leq 1$, then $\chi_{flex}(G) \leq k$.

χ_{flex} vs degeneracy

Theorem (K., Mathew, Mudrock, Pelsmajer (2022+))

Suppose n is a positive even integer, and $G = P_n \vee P_n$. If $p \in (0,1)$ and $k > \max\{2, n/2\}$ satisfy $(n/2)(1-p)^{k-2} + np^{k-1-n/2}(p + (k-n/2)(1-p)) \leq 1$, then $\chi_{flex}(G) \leq k$.

Corollary

Suppose n is a positive even integer satisfying $n \geq 50$. Then, $\chi_{flex}(P_n \vee P_n) \leq \lceil n/2 + \ln(n) \rceil$.

Since degeneracy of $P_n \vee P_n$ is $n+1$, it follows

Corollary

$\mathcal{F}(d)$ grows no faster than $d/2$ as $d \to \infty$.

$\chi_{flex}(G)$ vs $\chi_\ell(G)$

- Recall $\chi_\ell(G) \leq \chi_{flex}(G) \leq \Delta(G) + 1$.
- It is natural to ask whether $\chi_{flex}(G)$ can be bounded above by a function of $\chi_\ell(G)$.
  Question: Does there exist a function f such that for every graph G, $\chi_{flex}(G) \leq f(\chi_\ell(G))$?

$\chi_{flex}(G)$ vs $\chi_\ell(G)$

- Recall $\chi_\ell(G) \leq \chi_{flex}(G) \leq \Delta(G) + 1$.
- It is natural to ask whether $\chi_{flex}(G)$ can be bounded above by a function of $\chi_\ell(G)$.
  Question: Does there exist a function f such that for every graph G, $\chi_{flex}(G) \leq f(\chi_\ell(G))$?
- We show that there is no universal constant C such that $\chi_{flex}(G) \leq \chi_\ell(G) + C$.

$\chi_{flex}(G)$ vs $\chi_\ell(G)$

- We show that there is no universal constant C such that $\chi_{flex}(G) \le \chi_\ell(G) + C$.

Proposition (K., Mathew, Mudrock, Pelsmajer (2022+))

Suppose that $l \in \mathbb{N}$ and $t_0 = \sum_{i=0}^{l} \binom{2l+1}{2l+1-i}(2l)^i$.
(i) For $t \ge t_0$, $\chi_{flex}(K_{2l+1,t}) \ge 2l + 2$.
(ii) For $s \le t_0$, $\chi_\ell(K_{2l+1,s}) \le \lceil 3l/2 \rceil$ whenever $l \ge 100000$.

Corollary

Let $t_0 = \sum_{i=0}^{l} \binom{2l+1}{2l+1-i}(2l)^i$.
For each $l \ge 100000$, $\chi_{flex}(K_{2l+1,t}) \ge \frac{4}{3}\chi_\ell(K_{2l+1,t}) + \frac{4}{3}$.

χ_{flex} vs List packing

- An idea implicit in an earlier work (for k-trees):

Proposition (K., Mathew, Mudrock, Pelsmajer (2022+))

Suppose that G is a graph, L is a k-assignment for G, and there is a set $\mathcal{S}$ of mk proper L-colorings such that for each vertex $v \in V(G)$ and each color $c \in L(v)$, vertex v is colored by c in exactly m of the L-colorings of $\mathcal{S}$. Then, (G, L, r) is $1/k$-satisfiable for any request r of L.

χ_{flex} vs List packing

- An idea implicit in an earlier work (for k-trees):

Proposition (K., Mathew, Mudrock, Pelsmajer (2022+))

Suppose that G is a graph, L is a k-assignment for G, and there is a set $\mathcal{S}$ of mk proper L-colorings such that for each vertex $v \in V(G)$ and each color $c \in L(v)$, vertex v is colored by c in exactly m of the L-colorings of $\mathcal{S}$. Then, (G, L, r) is $1/k$-satisfiable for any request r of L.

- For a non-trivial tree T, it is easy to see that for any 2-assignment L, there exist 2 proper L-colorings that are distinct on each vertex. Here $m = 1$ and $k = 2$. Since $\rho(T) = 2$ and $\chi_{flex}(T) \geq \chi_\ell(T) = 2$, above Proposition implies $\chi_{flex}(T) = 2$.

χ_{flex} vs List packing

- List packing is a relatively new notion that was first suggested by Alon, Fellows, and Hare (1996), and formally defined in a recent paper of Cambie, Batenburg, Davies, Kang (2021+/ to appear in RSA 2023).

χ_{flex} vs List packing

- List packing is a relatively new notion that was first suggested by Alon, Fellows, and Hare (1996), and formally defined in a recent paper of Cambie, Batenburg, Davies, Kang (2021+/ to appear in RSA 2023).

- Let L be a list assignment for a graph G. An L-packing of G of size k is a set of proper L-colorings $\{f_1, \ldots, f_k\}$ of G such that $f_i(v) \neq f_j(v)$ whenever $i, j \in [k]$, $i \neq j$, and $v \in V(G)$. The list packing number of G, denoted $\chi_\ell^*(G)$, is the least k such that G has a proper L-packing of size k whenever L is a k-assignment for G.

- $\chi(G) \leq \chi_\ell(G) \leq \chi_\ell^*(G)$.

χ_{flex} vs List packing

Proposition (K., Mathew, Mudrock, Pelsmajer (2022+))

Suppose that G is a graph, L is a k-assignment for G, and there is a set $\mathcal{S}$ of mk proper L-colorings such that for each vertex $v \in V(G)$ and each color $c \in L(v)$, vertex v is colored by c in exactly m of the L-colorings of $\mathcal{S}$. Then, (G, L, r) is $1/k$-satisfiable for any request r of L.

Corollary (K., Mathew, Mudrock, Pelsmajer (2022+))

For any graph G, G is $(\chi_\ell^*(G), 1/\chi_\ell^*(G))$-flexible.

χ_{flex} vs List packing

- The list packing number of G, denoted $\chi_\ell^*(G)$, is the least k such that G has a proper L-packing of size k whenever L is a k-assignment for G.
  $\chi(G) \leq \chi_\ell(G) \leq \chi_\ell^*(G)$.

Corollary (K., Mathew, Mudrock, Pelsmajer (2022+))

For any graph G, G is $(\chi_\ell^*(G), 1/\chi_\ell^*(G))$-flexible.

χ_{flex} vs List packing

- The list packing number of G, denoted $\chi_\ell^*(G)$, is the least k such that G has a proper L-packing of size k whenever L is a k-assignment for G.
  $\chi(G) \leq \chi_\ell(G) \leq \chi_\ell^*(G)$.

Corollary (K., Mathew, Mudrock, Pelsmajer (2022+))

For any graph G, G is $(\chi_\ell^*(G), 1/\chi_\ell^*(G))$-flexible.

- In view of the Corollary above, and since $\chi_\ell(G) \leq \chi_{flex}(G)$, it is natural to ask whether $\chi_{flex}(G)$ can be bounded above by a function of $\chi_\ell^*(G)$. More ambitiously,
  Conjecture: For any graph G, $\chi_{flex}(G) \leq \chi_\ell^*(G)$.

χ_{flex} vs List packing

- Conjecture: For any graph G, $\chi_{flex}(G) \leq \chi_\ell^*(G)$.
- Cambie and Hämäläinen (2023+): $\chi_\ell^\star(K_{3,7}) = 3$, disprove this conjecture with help of K., Mathew, Mudrock, Pelsmajer (2022+): $\chi_{flex}(K_{3,7}) > 3$, and conjecture there are infinitely many counterexamples.

χ_{flex} vs List packing

- Conjecture: For any graph G, $\chi_{flex}(G) \leq \chi^*_\ell(G)$.

- Cambie and Hämäläinen (2023+): $\chi^\star_\ell(K_{3,7}) = 3$, disprove this conjecture with help of K., Mathew, Mudrock, Pelsmajer (2022+): $\chi_{flex}(K_{3,7}) > 3$, and conjecture there are infinitely many counterexamples.

- However, it follows from our Proposition and a result of Cambie, Batenburg, Davies, Kang (2021+) that: Every graph G on n vertices is $(\chi^*_\ell(G), 1/((5 + o(1))\rho(G)(\log n)^2))$-flexible where the $o(1)$ term tends to 0 as n tends to infinity.

χ_{flex} using List packing

Proposition (K., Mathew, Mudrock, Pelsmajer (2022+))

Suppose that G is a graph, L is a k-assignment for G, and there is a set $\mathcal{S}$ of mk proper L-colorings such that for each vertex $v \in V(G)$ and each color $c \in L(v)$, vertex v is colored by c in exactly m of the L-colorings of $\mathcal{S}$. Then, (G, L, r) is $1/k$-satisfiable for any request r of L.

Corollary (K., Mathew, Mudrock, Pelsmajer (2022+))

For any graph G, G is $(\chi_\ell^*(G), 1/\chi_\ell^*(G))$-flexible.

χ_{flex} using List packing

Corollary (K., Mathew, Mudrock, Pelsmajer (2022+))

For any graph G, G is $(\chi_\ell^*(G), 1/\chi_\ell^*(G))$*-flexible.*

- For a non-trivial tree T, it is easy to see that $(\chi_\ell^*(T) = 2$. Since $\rho(T) = 2$ and $\chi_{flex}(T) \geq \chi_\ell(T) = 2$, above Proposition implies $\chi_{flex}(T) = 2$.
- It follows from above Proposition and a result of Cambie, Batenburg, Davies, Kang (2021+) that:
  Every graph G on n vertices is $(\chi_\ell^*(G), 1/((5 + o(1))\rho(G)(\log n)^2))$-flexible where the $o(1)$ term tends to 0 as n tends to infinity.

χ_{flex} using List packing

- Using $\chi_\ell^*(P_n) = 2$ and our Proposition, we get

Proposition (K., Mathew, Mudrock, Pelsmajer (2022+))

The grid $P_n \square P_m$ is $(3, 1/3)$-flexible.

And, we are able to obtain the best possible result for the n-ladder $P_2 \square P_n$.

Proposition (K., Mathew, Mudrock, Pelsmajer (2022+))

Suppose $G = P_2 \square P_n$ with $n \geq 2$. Then, G is $(3, 1/2)$-flexible. Consequently, $\chi_{flex}(G) = 3$.

and, more generally

Proposition (K., Mathew, Mudrock, Pelsmajer (2022+))

Suppose G is (k, ϵ)-flexible. Then $G \square H$ is $(\max\{k, \Delta(H) + \chi_\ell(G)\}, \epsilon/\chi(H))$-flexible.

Thank You! Questions?

- Dvořák, Norin, and Postle (2019): Are all d-degenerate graphs, $(d+1, \epsilon(d))$-flexible for some $\epsilon(d) > 0$?
- For any d-degenerate graph G with a $(d+1)$-list assignment L and request r with domain D of size 1, show that (G, L, r) is 1-satisfiable.
- What are all the graphs G such that $\chi_{flex}(G) = \Delta(G) + 1$?
- Does there exist a graph G with degeneracy d satisfying $\chi_{flex}(G) > d + 1$?
- What is the asymptotic behavior of $\mathcal{F}(d) = \min\{\chi_{flex}(G) : \text{the degeneracy of } G \text{ is at least } d\}$ as $d \to \infty$? Linear or Logarithmic in d?
- Can $\chi_{flex}(G)$ can be bounded above by a function of $\chi_{\ell}(G)$?
- Can $\chi_{flex}(G)$ can be bounded above by a function of $\chi_{\ell}^{*}(G)$? Are there infinitely many counterexamples to $\chi_{flex}(G) \leq \chi_{\ell}^{*}(G)$?
- Let ϵ_G be the function that maps each $k \in \mathbb{N}$ to the largest ϵ such that G is (k, ϵ)-flexible. Clearly, $\epsilon_G(k) = a/b$ for some integers $0 \leq a \leq b \leq |V(G)|$ and $\epsilon_G(k) \leq 1/\rho(G)$. Study ϵ_G for various G.

Thank You! Questions?

- Dvořák, Norin, and Postle (2019): Are all d-degenerate graphs, $(d+1, \epsilon(d))$-flexible for some $\epsilon(d) > 0$?
- For any d-degenerate graph G with a $(d+1)$-list assignment L and request r with domain D of size 1, show that (G, L, r) is 1-satisfiable.
- What are all the graphs G such that $\chi_{flex}(G) = \Delta(G) + 1$?
- Does there exist a graph G with degeneracy d satisfying $\chi_{flex}(G) > d + 1$?
- What is the asymptotic behavior of $\mathcal{F}(d) = \min\{\chi_{flex}(G) : \text{the degeneracy of } G \text{ is at least } d\}$ as $d \to \infty$? Linear or Logarithmic in d?
- Can $\chi_{flex}(G)$ can be bounded above by a function of $\chi_\ell(G)$?
- Can $\chi_{flex}(G)$ can be bounded above by a function of $\chi_\ell^*(G)$? Are there infinitely many counterexamples to $\chi_{flex}(G) \leq \chi_\ell^*(G)$?
- Let ϵ_G be the function that maps each $k \in \mathbb{N}$ to the largest ϵ such that G is (k, ϵ)-flexible. Clearly, $\epsilon_G(k) = a/b$ for some integers $0 \leq a \leq b \leq |V(G)|$ and $\epsilon_G(k) \leq 1/\rho(G)$. Study ϵ_G for various G.