Friday, October 26, 2012

Prove that

\[
\left(1 + \frac{1}{4}\right) \left(1 + \frac{1}{8}\right) \cdots \left(1 + \frac{1}{2^n}\right) < 2,
\]

for any \( n \geq 2 \).

**Solution.** Note that

\[
\ln \left[ \left(1 + \frac{1}{4}\right) \left(1 + \frac{1}{8}\right) \cdots \left(1 + \frac{1}{2^n}\right) \right] = \ln \left(1 + \frac{1}{4}\right) + \ldots + \ln \left(1 + \frac{1}{2^n}\right) < \frac{1}{4} + \frac{1}{8} + \ldots + \frac{1}{2^n} + \ldots = \frac{1/4}{1 - 1/2} = \frac{1}{2},
\]

where we used the fact that \( \ln(1 + x) < x, \quad x > 0 \). The former follows directly from the fact that \( x - \ln(1 + x) \) is an increasing function (just take the derivative). Hence,

\[
\left(1 + \frac{1}{4}\right) \left(1 + \frac{1}{8}\right) \cdots \left(1 + \frac{1}{2^n}\right) < \sqrt{e} < 2.
\]

Good Luck! Have fun and enjoy Mathematics!