MATH 532: Linear Algebra
Chapter 4: Vector Spaces

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Outline

1. Spaces and Subspaces
2. Four Fundamental Subspaces
3. Linear Independence
4. Bases and Dimension
5. More About Rank
6. Classical Least Squares
7. Kriging as best linear unbiased predictor
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Spaces and Subspaces

While the discussion of vector spaces can be rather dry and abstract, they are an essential tool for describing the world we work in, and to understand many practically relevant consequences.

After all, linear algebra is pretty much the workhorse of modern applied mathematics.

Moreover, many concepts we discuss now for traditional “vectors” apply also to vector spaces of functions, which form the foundation of functional analysis.
Spaces and Subspaces

Vector Space

Definition

A set $\mathcal{V}$ of elements (vectors) is called a vector space (or linear space) over the scalar field $\mathcal{F}$ if

(A1) $x + y \in \mathcal{V}$ for any $x, y \in \mathcal{V}$ (closed under addition),

(M1) $\alpha x \in \mathcal{V}$ for every $\alpha \in \mathcal{F}$ and $x \in \mathcal{V}$ (closed under scalar multiplication),

(A2) $(x + y) + z = x + (y + z)$ for all $x, y, z \in \mathcal{V},$

(M2) $(\alpha \beta)x = \alpha(\beta x)$ for all $\alpha \beta \in \mathcal{F}$, $x \in \mathcal{V},$

(A3) $x + y = y + x$ for all $x, y \in \mathcal{V},$

(M3) $\alpha(x + y) = \alpha x + \alpha y$ for all $\alpha \in \mathcal{F}$, $x, y \in \mathcal{V},$

(A4) There exists a zero vector $0 \in \mathcal{V}$ such that $x + 0 = x$ for every $x \in \mathcal{V},$

(M4) $(\alpha + \beta)x = \alpha x + \beta x$ for all $\alpha, \beta \in \mathcal{F}$, $x \in \mathcal{V},$

(A5) For every $x \in \mathcal{V}$ there is a negative $(-x) \in \mathcal{V}$ such that $x + (-x) = 0,$

(M5) $1x = x$ for all $x \in \mathcal{V}.$
Examples of vector spaces

Spaces and Subspaces

\[ V = \mathbb{R}^m \text{ and } F = \mathbb{R} \text{ (traditional real vectors)} \]

\[ V = \mathbb{C}^m \text{ and } F = \mathbb{C} \text{ (traditional complex vectors)} \]

\[ V = \mathbb{R}^{m \times n} \text{ and } F = \mathbb{R} \text{ (real matrices)} \]

\[ V = \mathbb{C}^{m \times n} \text{ and } F = \mathbb{C} \text{ (complex matrices)} \]

But also

\[ V \text{ is polynomials of a certain degree with real coefficients, } F = \mathbb{R} \]

\[ V \text{ is continuous functions on an interval } [a, b], F = \mathbb{R} \]
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But also

- \( V \) is polynomials of a certain degree with real coefficients, \( F = \mathbb{R} \)
- \( V \) is continuous functions on an interval \([a, b]\), \( F = \mathbb{R} \)
Definition

Let $S$ be a nonempty subset of $V$. If $S$ is a vector space, then $S$ is called a subspace of $V$. 

Theorem

The subset $S \subseteq V$ is a subspace of $V$ if and only if $\alpha x + \beta y \in S$ for all $x, y \in S$, $\alpha, \beta \in F$. (1)

Remark

$Z = \{0\}$ is called the trivial subspace.
Subspaces

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A: The structure provided by the axioms (A1)–(A5), (M1)–(M5)
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Definition

Let $S$ be a nonempty subset of $V$. If $S$ is a vector space, then $S$ is called a **subspace** of $V$.

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**A:** The **structure** provided by the axioms (A1)–(A5), (M1)–(M5)

**Theorem**

The subset $S \subseteq V$ is a subspace of $V$ if and only if

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**Remark**

$\mathcal{Z} = \{0\}$ is called the **trivial subspace**.
Proof.

“⇒” : Clear, since we actually have

(1) ⇔ (A1) and (M1)
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\[(1) \iff (A1) \text{ and } (M1)\]

“⇐”: Only (A1), (A4), (A5) and (M1) need to be checked (why?).
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If \(x \in S\), then — using (M1) — \(-1x = -x \in S\), i.e., (A5) holds.
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If \(x \in S\), then — using (M1) — \(-1x = -x \in S\), i.e., (A5) holds.

Using (A1), \(x + (-x) = 0 \in S\), so that (A4) holds.
Definition

Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_r\} \subseteq \mathcal{V}$. The span of $S$ is

$$\text{span}(S) = \left\{ \sum_{i=1}^{r} \alpha_i \mathbf{v}_i : \alpha_i \in \mathcal{F} \right\}.$$
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- $\text{span}(S)$ contains all possible linear combinations of vectors in $S$.
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Definition
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Example (Geometric interpretation)
1. If \( S = \{\mathbf{v}_1\} \subseteq \mathbb{R}^3 \), then \( \text{span}(S) \) is
Definition
Let $S = \{v_1, \ldots, v_r\} \subseteq V$. The span of $S$ is

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Example (Geometric interpretation)
1. If $S = \{v_1\} \subseteq \mathbb{R}^3$, then $\text{span}(S)$ is the line through the origin with direction $v_1$. 
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2. If \( S = \{ \mathbf{v}_1, \mathbf{v}_2 : \mathbf{v}_1 \neq \alpha \mathbf{v}_2, \alpha \neq 0 \} \subseteq \mathbb{R}^3 \), then \( \text{span}(S) \) is
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Example (Geometric interpretation)

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Definition

Let $\mathcal{S} = \{\mathbf{v}_1, \ldots, \mathbf{v}_r\} \subseteq \mathcal{V}$. If $\text{span} \mathcal{S} = \mathcal{V}$ then $\mathcal{S}$ is called a spanning set for $\mathcal{V}$.

Remark

A spanning set is sometimes referred to as a (finite) frame. A spanning set is not the same as a basis since the spanning set may include redundancies.

Example

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix},
\begin{bmatrix}
2 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & 2 \\
\end{bmatrix}
\]

is a spanning set for $\mathbb{R}^3$.

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\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}
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- $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ is a spanning set for $\mathbb{R}^3$.
- $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} \right\}$ is also a spanning set for $\mathbb{R}^3$. 
Spaces and Subspaces

Connection to linear systems

Theorem

Let \( S = \{a_1, a_2, \ldots, a_n\} \) be the set of columns of an \( m \times n \) matrix \( A \).

\( \text{span}(S) = \mathbb{R}^m \) if and only if for every \( b \in \mathbb{R}^m \) there exists an \( x \in \mathbb{R}^n \) such that \( Ax = b \) (i.e., if and only if \( Ax = b \) is consistent for every \( b \in \mathbb{R}^m \)).
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Proof.

By definition, \( S \) is a spanning set for \( \mathbb{R}^m \) if and only if for every \( b \in \mathbb{R}^m \) there exist \( \alpha_1, \ldots, \alpha_n \in \mathbb{R} \) such that

\[
b = \alpha_1 a_1 + \ldots + \alpha_n a_n = Ax,
\]

where \( A = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \end{pmatrix} \) and \( x = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} \).
Remark

The sum

\[ X + Y = \{ x + y : x \in X, y \in Y \} \]

is a subspace of \( \mathcal{V} \) provided \( X \) and \( Y \) are subspaces.
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\[ \mathcal{X} + \mathcal{Y} = \{ \mathbf{x} + \mathbf{y} : \mathbf{x} \in \mathcal{X}, \mathbf{y} \in \mathcal{Y} \} \]

is a subspace of \( \mathcal{V} \) provided \( \mathcal{X} \) and \( \mathcal{Y} \) are subspaces.

If \( S_\mathcal{X} \) and \( S_\mathcal{Y} \) are spanning sets for \( \mathcal{X} \) and \( \mathcal{Y} \), respectively, then \( S_\mathcal{X} \cup S_\mathcal{Y} \)

is a spanning set for \( \mathcal{X} + \mathcal{Y} \).
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Four Fundamental Subspaces

Recall that a linear function $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ satisfies

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y) \quad \forall \alpha, \beta \in \mathbb{R}, \; x, y \in \mathbb{R}^n.$$
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Example
Let \( A \) be a real \( m \times n \) matrix and
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    f(x) = Ax, \quad x \in \mathbb{R}^n.
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The function \( f \) is linear
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Example

Let $A$ be a real $m \times n$ matrix and

\[ f(x) = Ax, \quad x \in \mathbb{R}^n. \]

The function $f$ is linear since $A(\alpha x + \beta y) = \alpha Ax + \beta Ay$. Moreover, the \textit{range} of $f$, \[ \mathcal{R}(f) = \{ Ax : x \in \mathbb{R}^n \} \subseteq \mathbb{R}^m, \]

is a \textit{subspace} of $\mathbb{R}^m$ since
Four Fundamental Subspaces

Recall that a linear function $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ satisfies

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y) \quad \forall \alpha, \beta \in \mathbb{R}, \ x, y \in \mathbb{R}^n.$$  

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$$f(x) = Ax, \quad x \in \mathbb{R}^n.$$  

The function $f$ is linear since $A(\alpha x + \beta y) = \alpha Ax + \beta Ay$. Moreover, the range of $f$,

$$\mathcal{R}(f) = \{Ax : \ x \in \mathbb{R}^n\} \subseteq \mathbb{R}^m,$$

is a subspace of $\mathbb{R}^m$ since for all $\alpha, \beta \in \mathbb{R}$ and $x, y \in \mathbb{R}^n$

$$\alpha(Ax) + \beta(Ay) = A(\alpha x + \beta y) \in \mathcal{R}(f).$$
Remark

For the situation in this example we can also use the terminology range of $A$ (or image of $A$), i.e.,

$$R(A) = \{Ax : x \in \mathbb{R}^n\} \subseteq \mathbb{R}^m$$
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Similarly,

$$R(A^T) = \left\{A^Ty : y \in \mathbb{R}^m\right\} \subseteq \mathbb{R}^n$$

is called the range of $A^T$. 
Column space and row space

Since

\[ Ax = \alpha_1 a_1 + \ldots + \alpha_n a_n, \]

we have \( R(A) = \text{span}\{a_1, \ldots, a_n\} \), i.e.,

\[ R(A) \text{ is the column space of } A. \]
Column space and row space

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Similarly,

\[ R(A^T) \text{ is the row space of } A. \]
Example

Consider

\[
A = \begin{pmatrix}
1 & 2 & 3 \\
4 & 5 & 6 \\
7 & 8 & 9
\end{pmatrix}
\]

By definition

- the columns of A span \( R(A) \), i.e., they form a spanning set of \( R(A) \),
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- the columns of $A$ span $R(A)$, i.e., they form a spanning set of $R(A)$,
- the rows of $A$ span $R(A^T)$, i.e., they form a spanning set of $R(A^T)$.

However, since

$$(A)_{3}^* = 2(A)_{2}^* - (A)_{1}^* \quad \text{and} \quad (A)_{3}^* = 2(A)_{2}^* - (A)_{1}^*$$

we also have

- $R(A) = \ldots$
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we also have

- $R(A) = \text{span}\{(A)_{1^\ast}, (A)_{2^\ast}\}$
- $R(A^T) =$
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- the columns of \( A \) span \( R(A) \), i.e., they form a spanning set of \( R(A) \),
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(A)_{3*} = 2(A)_{2*} - (A)_{1*} \quad \text{and} \quad (A)_{3*} = 2(A)_{2*} - (A)_{1*}
\]

we also have

- \( R(A) = \text{span}\{(A)_{1*}, (A)_{2*}\} \)
- \( R(A^T) = \text{span}\{(A)_{1*}, (A)_{2*}\} \)
In general, how do we find such minimal spanning sets as in the previous example?

An important tool is

**Lemma**

*Let \( A, B \) be \( m \times n \) matrices. Then*

\[
(1) \quad R(A^T) = R(B^T) \iff A^{\text{row}} \sim B \quad (\iff E_A = E_B).
\]

\[
(2) \quad R(A) = R(B) \iff A^{\text{col}} \sim B \quad (\iff E_{A^T} = E_{B^T}).
\]
Proof

1. “⇐”: Assume $A \xrightarrow{\text{row}} B$, i.e., there exists a nonsingular matrix $P$ such that

$$PA = B \iff A^T P^T = B^T.$$
Proof

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Now $a \in R(A^T) \iff a = A^T y$ for some $y$. 

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We rewrite this as

$$a = A^TP^TP^{-T}y$$
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$$PA = B \iff A^T P^T = B^T.$$ 

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$$a = A^T P^T P^{-T} y = B^T.$$
Proof

“⇐”: Assume \( A \overset{\text{row}}{\sim} B \), i.e., there exists a nonsingular matrix \( P \) such that

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PA = B \iff A^T P^T = B^T.
\]

Now \( a \in R(A^T) \iff a = A^T y \) for some \( y \).

We rewrite this as

\[
a = A^T P^T P^{-T} y = B^T x \quad \text{for} \quad x = P^{-T} y.
\]
Proof

1. “⇐”: Assume $A \overset{\text{row}}{\sim} B$, i.e., there exists a nonsingular matrix $P$ such that

$$PA = B \iff A^T P^T = B^T.$$ 

Now $a \in R(A^T) \iff a = A^T y$ for some $y$. We rewrite this as

$$a = A^T P^T P^{-T} y = B^T x \iff a = B^T x \text{ for } x = P^{-T} y \iff a \in R(B^T).$$
(cont.)

“⇒”: Assume \( R(A^T) = R(B^T) \), i.e.,

\[
\text{span}\{(A)_1^*, \ldots, (A)_{m^*}\} = \text{span}\{(B)_1^*, \ldots, (B)_{m^*}\},
\]
"⇒": Assume $R(A^T) = R(B^T)$, i.e.,

$$\text{span}\{(A)_1^*, \ldots, (A)_{m^*}\} = \text{span}\{(B)_1^*, \ldots, (B)_{m^*}\},$$

i.e., the rows of A are linear combinations of rows of B and vice versa.
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Now apply row operations to $A$ (all collected in $P$) to obtain

$$PA = B, \text{ i.e., } A \sim B.$$
(cont.)

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i.e., the rows of $A$ are linear combinations of rows of $B$ and vice versa.

Now apply row operations to $A$ (all collected in $P$) to obtain

$$PA = B, \quad \text{i.e., } A \sim \text{ row } B.$$

Let $A = A^T$ and $B = B^T$ in (1).
Theorem

Let $A$ be an $m \times n$ matrix and $U$ any row echelon form obtained from $A$. Then

1. $R(A^T) = \text{span of nonzero rows of } U$.
2. $R(A) = \text{span of basic columns of } A$. 
Four Fundamental Subspaces

Theorem

Let $A$ be an $m \times n$ matrix and $U$ any row echelon form obtained from $A$. Then

1. $R(A^T) = \text{span of nonzero rows of } U$.
2. $R(A) = \text{span of basic columns of } A$.

Remark

Later we will see that any minimal span of the columns of $A$ forms a basis for $R(A)$. 
Proof

1. This follows from (1) in the Lemma since $A \sim U$. 
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1. This follows from (1) in the Lemma since $A \sim U$.

2. Assume the columns of $A$ are permuted (with a matrix $Q_1$) such that

$$AQ_1 = (B \ N),$$

where $B$ contains the basic columns, and $N$ the nonbasic columns.
Proof

1. This follows from (1) in the Lemma since $A \overset{\text{row}}{\sim} U$.

2. Assume the columns of $A$ are permuted (with a matrix $Q_1$) such that

$$AQ_1 = (B \quad N),$$

where $B$ contains the basic columns, and $N$ the nonbasic columns.

By definition, the nonbasic columns are linear combinations of the basic columns, i.e., there exists a nonsingular $Q_2$ such that

$$(B \quad N) Q_2 = (B \quad O),$$

where $O$ is a zero matrix.
Putting this together, we have

\[ AQ_1 Q_2 = (B \ O) , \]

(2) in the Lemma says that

\[ R(A) = \text{span}\{B^*1, \ldots, B^*r\} , \]

where \( r = \text{rank}(A) \). □
(cont.)
Putting this together, we have

\[ A Q_1 Q_2 = (B \quad O), \]

so that \( A \overset{\text{col}}{\sim} (B \quad O). \)
Putting this together, we have

\[ A Q_1 Q_2 = (B \ O), \]

so that \( A^{\text{col}} \sim (B \ O). \)

(2) in the Lemma says that

\[ R(A) = \text{span}\{B_1^*, \ldots, B_r^*\}, \]

where \( r = \text{rank}(A). \)
So far, we have two of the four fundamental subspaces:

\[ R(A) \quad \text{and} \quad R(A^T). \]
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Third fundamental subspace: \[ N(A) = \{ x : Ax = 0 \} \subseteq \mathbb{R}^n, \]

\[ N(A) \] is the nullspace of \( A \)

(also called the kernel of \( A \))
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Fourth fundamental subspace: \( N(A^T) = \{ y : A^T y = 0 \} \subseteq \mathbb{R}^m, \]

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Remark

\( N(A) \) is a linear space, i.e., a subspace of \( \mathbb{R}^n \).
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Remark

\( N(A) \) is a linear space, i.e., a subspace of \( \mathbb{R}^n. \)

To see this, assume \( x, y \in N(A), \) i.e., \( Ax = Ay = 0. \)
So far, we have two of the four fundamental subspaces:

\[ R(A) \text{ and } R(A^T). \]

Third fundamental subspace: \( N(A) = \{ \mathbf{x} : A\mathbf{x} = \mathbf{0} \} \subseteq \mathbb{R}^n, \)

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Fourth fundamental subspace: \( N(A^T) = \{ \mathbf{y} : A^T\mathbf{y} = \mathbf{0} \} \subseteq \mathbb{R}^m, \)

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Remark

\( N(A) \text{ is a linear space, i.e., a subspace of } \mathbb{R}^n. \)

To see this, assume \( \mathbf{x}, \mathbf{y} \in N(A), \text{ i.e., } A\mathbf{x} = A\mathbf{y} = \mathbf{0}. \)

Then

\[ A(\alpha \mathbf{x} + \beta \mathbf{y}) = \alpha A\mathbf{x} + \beta A\mathbf{y} = \mathbf{0}, \]

so that \( \alpha \mathbf{x} + \beta \mathbf{y} \in N(A). \)
How to find a (minimal) spanning set for $N(A)$

Find a row echelon form $U$ of $A$ and solve $Ux = 0$.

**Example**

We can compute $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$ $\xrightarrow{\text{U}}$ $U = \begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{pmatrix}$.
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Find a row echelon form $U$ of $A$ and solve $U\mathbf{x} = \mathbf{0}$.

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So that $U\mathbf{x} = \mathbf{0}$ \[\implies\] \[
\begin{cases}
  x_2 = -2x_3 \\
  x_1 = -2x_2 - 3x_3 = x_3,
\end{cases}
\]
How to find a (minimal) spanning set for $N(A)$

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x_1 &= -2x_2 - 3x_3 = x_3
\end{align*}

or

\[
\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_3 \\ -2x_3 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}.
\]
How to find a (minimal) spanning set for $N(A)$

Find a row echelon form $U$ of $A$ and solve $Ux = 0$.

Example

We can compute $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \quad \rightarrow \quad U = \begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{pmatrix}$.

So that $Ux = 0 \quad \Rightarrow \quad \begin{cases} x_2 = -2x_3 \\ x_1 = -2x_2 - 3x_3 = x_3 \end{cases}$, or

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_3 \\ -2x_3 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}.$$ 

Therefore

$$N(A) = \text{span} \left\{ \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \right\}.$$
Remark

We will see later that — as in the example — if \( \text{rank}(A) = r \), then \( N(A) \) is spanned by \( n - r \) vectors.
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Theorem

Let \( A \) be an \( m \times n \) matrix. Then

1. \( N(A) = \{0\} \iff \text{rank}(A) = n \).
2. \( N(A^T) = \{0\} \iff \text{rank}(A) = m \).
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Proof.

1. We know \( \text{rank}(A) = n \iff A x = 0, \)
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Proof.

1. We know \( \text{rank}(A) = n \iff Ax = 0 \), but that implies \( x = 0 \).
2. Repeat (1) with \( A = A^T \) and use \( \text{rank}(A^T) = \text{rank}(A) \).
How to find a spanning set of $N(A^T)$

**Theorem**

Let $A$ be an $m \times n$ matrix with $\text{rank}(A) = r$, and let $P$ be a nonsingular matrix so that $PA = U$ (row echelon form). Then the last $m - r$ rows of $P$ span $N(A^T)$. 
How to find a spanning set of $N(A^T)$

**Theorem**

Let $A$ be an $m \times n$ matrix with $\text{rank}(A) = r$, and let $P$ be a nonsingular matrix so that $PA = U$ (row echelon form). Then the last $m - r$ rows of $P$ span $N(A^T)$.

**Remark**

We will later see that this spanning set is also a basis for $N(A^T)$. 
Proof

Partition \( P \) as \( P = \begin{pmatrix} P_1 \\ P_2 \end{pmatrix} \), where \( P_1 \) is \( r \times m \) and \( P_2 \) is \( m - r \times m \).
Proof

Partition $P$ as $P = \begin{pmatrix} P_1 \\ P_2 \end{pmatrix}$, where $P_1$ is $r \times m$ and $P_2$ is $m - r \times m$.

The claim of the theorem implies that we should show that $R(P_2^T) = N(A^T)$.
Proof

Partition P as $P = \begin{pmatrix} P_1 \\ P_2 \end{pmatrix}$, where $P_1$ is $r \times m$ and $P_2$ is $m - r \times m$.

The claim of the theorem implies that we should show that $R(P_2^T) = N(A^T)$.

We do this in two parts:

1. Show that $R(P_2^T) \subseteq N(A^T)$.
2. Show that $N(A^T) \subseteq R(P_2^T)$. 

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Partition $U_{m \times n} = \begin{pmatrix} C \\ O \end{pmatrix}$ with $C \in \mathbb{R}^{r \times n}$ and $O \in \mathbb{R}^{m-r \times n}$ (a zero matrix).

Then

$$PA = U \iff \begin{pmatrix} P_1 \\ P_2 \end{pmatrix} A = \begin{pmatrix} C \\ O \end{pmatrix}$$
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Then

$$PA = U \iff \begin{pmatrix} P_1 \\ P_2 \end{pmatrix} A = \begin{pmatrix} C \\ O \end{pmatrix} \implies P_2 A = O.$$
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Then

$$PA = U \iff \begin{pmatrix} P_1 \\ P_2 \end{pmatrix} A = \begin{pmatrix} C \\ O \end{pmatrix} \implies P_2 A = O.$$ 

This also means that

$$A^T P_2^T = O^T,$$
Partition $U_{m \times n} = \begin{pmatrix} C \\ O \end{pmatrix}$ with $C \in \mathbb{R}^{r \times n}$ and $O \in \mathbb{R}^{m-r \times n}$ (a zero matrix).

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This also means that

$$A^T P_2^T = O^T,$$

i.e., every column of $P_2^T$ is in $N(A^T)$ so that $R(P_2^T) \subseteq N(A^T)$. 
Now, show $N(A^T) \subseteq R(P_2^T)$. 

By definition, $y \in N(A^T) = \iff A^T y = 0$. Since $P A = U = \iff A = P^{-1} U$, and so $0^T = y^T P^{-1} U = y^T P^{-1} (C_0)$ or $0^T = y^T Q_1 C_2$, where $P^{-1} = (Q_1 \cdots Q_r m \times r Q_2 m \times m - r)$. 

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Now, show $N(A^T) \subseteq R(P_2^T)$.

We assume $y \in N(A^T)$ and show that then $y \in R(P_2^T)$. 

By definition, $y \in N(A^T) \iff A^T y = 0 \iff y^T A = 0^T$.

Since $P A = U \Rightarrow A = P^{-1} U$, and so $0^T = y^T P^{-1} U$ or $0^T = y^T Q_1 C$,
Now, show $N(A^T) \subseteq R(P_2^T)$.

We assume $y \in N(A^T)$ and show that then $y \in R(P_2^T)$.

By definition,

$$y \in N(A^T) \implies A^T y = 0 \iff y^T A = 0^T.$$
Now, show \( N(A^T) \subseteq R(P_2^T) \).

We assume \( y \in N(A^T) \) and show that then \( y \in R(P_2^T) \).

By definition,

\[
y \in N(A^T) \implies A^T y = 0 \iff y^T A = 0^T.
\]

Since \( PA = U \implies A = P^{-1} U \), and so

\[
0^T = y^T P^{-1} U = y^T P^{-1} \begin{pmatrix} C \\ O \end{pmatrix}
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Now, show $N(A^T) \subseteq R(P_2^T)$.

We assume $y \in N(A^T)$ and show that then $y \in R(P_2^T)$.

By definition,

$$y \in N(A^T) \implies A^T y = 0 \iff y^T A = 0^T.$$  

Since $PA = U \implies A = P^{-1}U$, and so

$$0^T = y^T P^{-1} U = y^T P^{-1} \begin{pmatrix} C \\ O \end{pmatrix}$$

or

$$0^T = y^T Q_1 C, \quad \text{where} \quad P^{-1} = \begin{pmatrix} Q_1 & Q_2 \\ m \times r & m \times m-r \end{pmatrix}.$$
(cont.)

However, since rank(C) = r and C is $m \times n$ we get (using $m = r$ in our earlier theorem)

$$N(C^T) = \{0\}$$

and therefore $y^T Q_1 = 0^T$. 
(cont.)
However, since $\text{rank}(C) = r$ and $C$ is $m \times n$ we get (using $m = r$ in our earlier theorem)

$$N(C^T) = \{0\}$$

and therefore $y^TQ_1 = 0^T$.

Obviously, this implies that we also have

$$y^TQ_1P_1 = 0^T$$  \hspace{1cm} (2)
(cont.)

Now \( P = \begin{pmatrix} P_1 \\ P_2 \end{pmatrix} \) and \( P^{-1} = \begin{pmatrix} Q_1 & Q_2 \end{pmatrix} \) so that

\[
I = P^{-1}P
\]
(cont.)

Now \( P = \begin{pmatrix} P_1 \\ P_2 \end{pmatrix} \) and \( P^{-1} = (Q_1 \quad Q_2) \) so that

\[
I = P^{-1}P = Q_1P_1 + Q_2P_2
\]
(cont.)

Now \( P = \begin{pmatrix} P_1 \\ P_2 \end{pmatrix} \) and \( P^{-1} = \begin{pmatrix} Q_1 & Q_2 \end{pmatrix} \) so that

\[
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\]

or

\[
Q_1P_1 = I - Q_2P_2. \tag{3}
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Now \( P = \begin{pmatrix} P_1 \\ P_2 \end{pmatrix} \) and \( P^{-1} = (Q_1 \quad Q_2) \) so that

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(3)

Now we insert (3) into (2) and get
(cont.)

Now \( P = \begin{pmatrix} P_1 \\ P_2 \end{pmatrix} \) and \( P^{-1} = \begin{pmatrix} Q_1 & Q_2 \end{pmatrix} \) so that

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or

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Q_1P_1 = I - Q_2P_2. \tag{3}
\]

Now we insert (3) into (2) and get

\[
y^T(I - Q_2P_2)
\]
Now \( P = \begin{pmatrix} P_1 \\ P_2 \end{pmatrix} \) and \( P^{-1} = (Q_1 \quad Q_2) \) so that

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Q_1P_1 = I - Q_2P_2. \tag{3}
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Now we insert (3) into (2) and get

\[
y^T(I - Q_2P_2) = 0^T \iff y^T = y^TQ_2P_2
\]
Now \( P = \begin{pmatrix} P_1 \\ P_2 \end{pmatrix} \) and \( P^{-1} = \begin{pmatrix} Q_1 & Q_2 \end{pmatrix} \) so that

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Now \( P = \begin{pmatrix} P_1 \\ P_2 \end{pmatrix} \) and \( P^{-1} = (Q_1 \quad Q_2) \) so that

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\]

or

\[
Q_1 P_1 = I - Q_2 P_2. \tag{3}
\]

Now we insert (3) into (2) and get

\[
y^T (I - Q_2 P_2) = 0^T \iff y^T = y^T Q_2 P_2 = z^T.
\]

Therefore \( y \in R(P_2^T) \).
Finally,

**Theorem**

*Let $A, B$ be $m \times n$ matrices.*

1. $N(A) = N(B) \iff A \overset{\text{row}}{\sim} B$.  
2. $N(A^T) = N(B^T) \iff A \overset{\text{col}}{\sim} B$.

**Proof.**

See [Mey00, Section 4.2].
Outline

1. Spaces and Subspaces
2. Four Fundamental Subspaces
3. Linear Independence
4. Bases and Dimension
5. More About Rank
6. Classical Least Squares
7. Kriging as best linear unbiased predictor
Linear Independence

Definition
A set of vectors $S = \{ \mathbf{v}_1, \ldots, \mathbf{v}_n \}$ is called \textit{linearly independent} if

$$\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \ldots + \alpha_n \mathbf{v}_n = 0 \implies \alpha_1 = \alpha_2 = \ldots = \alpha_n = 0.$$ 

Otherwise $S$ is \textit{linearly dependent}.

Remark
\begin{center}Linear independence is a property of a set, not of vectors.\end{center}
Example

Is \( S = \left\{ \begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \\ 8 \end{pmatrix}, \begin{pmatrix} 3 \\ 6 \\ 9 \end{pmatrix} \right\} \) linearly independent?
Linear Independence

Example

Is \( S = \left\{ \begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \\ 8 \end{pmatrix}, \begin{pmatrix} 3 \\ 6 \\ 9 \end{pmatrix} \right\} \) linearly independent?

Consider

\[ \alpha_1 \begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix} + \alpha_2 \begin{pmatrix} 2 \\ 5 \\ 8 \end{pmatrix} + \alpha_3 \begin{pmatrix} 3 \\ 6 \\ 9 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \]
Example

Is \( S = \left\{ \begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \\ 8 \end{pmatrix}, \begin{pmatrix} 3 \\ 6 \\ 9 \end{pmatrix} \right\} \) linearly independent?

Consider

\[ \alpha_1 \begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix} + \alpha_2 \begin{pmatrix} 2 \\ 5 \\ 8 \end{pmatrix} + \alpha_3 \begin{pmatrix} 3 \\ 6 \\ 9 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \]

\[ \iff \quad \mathbf{A} \mathbf{x} = \mathbf{0}, \quad \text{where} \quad \mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} \]
Example ((cont.))

Since

\[ A \overset{\text{row}}{\sim} E_A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \]

we know that \( N(A) \) is nontrivial, i.e., the system \( Ax = 0 \) has a nonzero solution, and therefore \( S \) is linearly dependent.
More generally,

**Theorem**

Let $A$ be an $m \times n$ matrix.

1. The columns of $A$ are linearly independent if and only if
   \[ N(A) = \{0\} \iff \text{rank}(A) = n. \]

2. The rows of $A$ are linearly independent if and only if
   \[ N(A^T) = \{0\} \iff \text{rank}(A) = m. \]

**Proof.**

See [Mey00, Section 4.3].
Definition
A square matrix $A$ is called **diagonally dominant** if

$$|a_{ii}| > \sum_{\substack{j=1 \atop j \neq i}}^{n} |a_{ij}|, \quad i = 1, \ldots, n.$$
Definition
A square matrix $A$ is called **diagonally dominant** if

$$|a_{ii}| > \sum_{j=1 \atop j \neq i}^{n} |a_{ij}|, \quad i = 1, \ldots, n.$$ 

Remark
- Aside from being nonsingular (see next slide), diagonally dominant matrices are important since they ensure that **Gaussian elimination will succeed without pivoting.**
- Also, diagonally dominance ensures convergence of certain iterative solvers (more later).
Theorem

Let $A$ be an $n \times n$ matrix. If $A$ is diagonally dominant then $A$ is nonsingular.
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Proof

We will show that $N(A) = \{0\}$ since then we know that $\text{rank}(A) = n$ and $A$ is nonsingular.

We will do this with a proof by contradiction.
Theorem

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We will do this with a proof by contradiction.

We assume that there exists an $x(\neq 0) \in N(A)$ and we will conclude that $A$ cannot be diagonally dominant.
(cont.)
If \( \mathbf{x} \in N(A) \) then \( A\mathbf{x} = 0 \).
If \( x \in N(A) \) then \( Ax = 0 \).

Now we take \( k \) so that \( x_k \) is the maximum (in absolute value) component of \( x \) and consider

\[
A_{k^*} x = 0.
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If \( \mathbf{x} \in N(A) \) then \( A\mathbf{x} = \mathbf{0} \).

Now we take \( k \) so that \( x_k \) is the maximum (in absolute value) component of \( \mathbf{x} \) and consider

\[ A_k \mathbf{x} = \mathbf{0}. \]

We can rewrite this as

\[
\sum_{j=1}^{n} a_{kj} x_j = 0 \quad \iff
\]

\[
\sum_{j=1}^{n} a_{kj} x_j = 0
\]
(cont.)

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Now we take \( k \) so that \( x_k \) is the maximum (in absolute value) component of \( x \) and consider

\[
A_k^* x = 0.
\]

We can rewrite this as

\[
\sum_{j=1}^{n} a_{kj} x_j = 0 \iff a_{kk} x_k = - \sum_{j=1 \atop j \neq k}^{n} a_{kj} x_j.
\]
Linear Independence (cont.)

Now we take absolute values:

\[ |a_{kk}x_k| = \left| \sum_{j=1}^{n} a_{kj}x_j \right| \]

Finally, dividing both sides by \(|x_k|\) yields

\[ |a_{kk}| \leq \left| \sum_{j=1}^{n} a_{kj}x_j \right| \]

which shows that A cannot be diagonally dominant (which is a contradiction since A was assumed to be diagonally dominant). □
(cont.)

Now we take absolute values:

$$|a_{kk}x_k| = \left| \sum_{j=1 \atop j \neq k}^n a_{kj}x_j \right| \leq \sum_{j=1 \atop j \neq k}^n |a_{kj}| |x_j|$$

Finally, dividing both sides by $|x_k|$ yields

$$\left| a_{kk} \right| \leq \sum_{j=1 \atop j \neq k}^n |a_{kj}| |x_j|,$$

which shows that $A$ cannot be diagonally dominant (which is a contradiction since $A$ was assumed to be diagonally dominant). \(\square\)
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\leq \left( \sum_{j=1 \atop j \neq k}^{n} |a_{kj}| \right) \max. \text{ component} \sum_{j=1 \atop j \neq k}^{n} |x_j|
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Now we take absolute values:

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Finally, dividing both sides by \(|x_k|\) yields

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which shows that A cannot be diagonally dominant (which is a contradiction since A was assumed to be diagonally dominant). \( \square \)
Example

Consider $m$ real numbers $x_1, \ldots, x_m$ such that $x_i \neq x_j$, $i \neq j$. Show that the columns of the Vandermonde matrix

$$V = \begin{pmatrix}
1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\
1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\
& \vdots & \ddots & \vdots & \vdots \\
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form a linearly independent set provided $n \leq m$. 
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form a linearly independent set provided $n \leq m$.
From above, the columns of $V$ are linearly independent if and only if $N(V) = \{0\}$

$$\iff Vz = 0 \implies z = 0, \quad z = \begin{pmatrix}
\alpha_0 \\
\vdots \\
\alpha_{n-1}
\end{pmatrix}.$$
Example (cont.)
Now $Vz = 0$ if and only if

$$\alpha_0 + \alpha_1 x_i + \alpha_2 x_i^2 + \ldots + \alpha_{n-1} x_i^{n-1} = 0, \quad i = 1, \ldots, m.$$
Example
(cont.)
Now $Vz = 0$ if and only if

$$\alpha_0 + \alpha_1 x_i + \alpha_2 x_i^2 + \ldots + \alpha_{n-1} x_i^{n-1} = 0, \quad i = 1, \ldots, m.$$ 

In other words, $x_1, x_2, \ldots, x_m$ are all (distinct) roots of

$$p(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \ldots + \alpha_{n-1} x^{n-1}.$$ 

This is a polynomial of degree at most $n - 1$.

It can have $m$ distinct roots only if $m \leq n - 1$.

Otherwise, $p$ is the zero polynomial, i.e., $\alpha_0 = \alpha_1 = \ldots = \alpha_{n-1} = 0$, so that the columns of $V$ are linearly dependent.
The example implies that in the special case $m = n$ there is a unique polynomial of degree (at most) $m - 1$ that interpolates the data 
$\{(x_1, y_1), (x_2, y_2), \ldots, (x_m, y_m)\} \subset \mathbb{R}^2$. 

We see this by writing the polynomial in the form
$$\ell(t) = \alpha_0 + \alpha_1 t + \alpha_2 t^2 + \ldots + \alpha_{m-1} t^{m-1}.$$ 

Then, interpolation of the data implies
$$\ell(x_i) = y_i, \quad i = 1, \ldots, m.$$ 

or
$$\begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{m-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{m-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_m & x_m^2 & \cdots & x_m^{m-1} \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{m-1} \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}.$$ 

Since the columns of $V$ are linearly independent it is nonsingular, and the coefficients $\alpha_0, \ldots, \alpha_{m-1}$ are uniquely determined.
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Since the columns of $V$ are linearly independent it is nonsingular, and the coefficients $\alpha_0, \ldots, \alpha_{m-1}$ are uniquely determined.
In fact,

$$\ell(t) = \sum_{i=1}^{m} y_i L_i(t)$$  \hspace{1cm} \text{(Lagrange interpolation polynomial)}

with

$$L_i(t) = \prod_{k=1}^{m} \frac{(t - x_k)}{\prod_{k=1, k \neq i}^{m} (x_i - x_k)}$$  \hspace{1cm} \text{(Lagrange functions)}.$$
In fact,

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\[ L_i(t) = \prod_{\substack{k=1 \atop k \neq i}}^{m} (t - x_k) / \prod_{\substack{k=1 \atop k \neq i}}^{m} (x_i - x_k) \]  

(Lagrange functions).

To verify (4) we note that the degree of \( \ell \) is \( m - 1 \) (since each \( L_i \) is of degree \( m - 1 \)) and

\[ L_i(x_j) = \delta_{ij}, \quad i, j = 1, \ldots, m, \]

so that

\[ \ell(x_j) = \sum_{i=1}^{m} y_i L_i(x_j) = y_j, \quad j = 1, \ldots, m. \]
Theorem

Let $S = \{u_1, u_2 \ldots, u_n\} \subseteq \mathcal{V}$ be nonempty. Then

1. If $S$ contains a linearly dependent subset, then $S$ is linearly dependent.

2. If $S$ is linearly independent, then every subset of $S$ is also linearly independent.

3. If $S$ is linearly independent and if $v \in \mathcal{V}$, then $S_{\text{ext}} = S \cup \{v\}$ is linearly independent if and only if $v \notin \text{span}(S)$.

4. If $S \subseteq \mathbb{R}^m$ and $n > m$, then $S$ must be linearly dependent.
Proof

If $S$ contains a linearly dependent subset, $\{u_1, \ldots, u_k\}$ say, then there exist nontrivial coefficients $\alpha_1, \ldots, \alpha_k$ such that

$$\alpha_1 u_1 + \ldots + \alpha_k u_k = 0.$$ 

Clearly, then

$$\alpha_1 u_1 + \ldots + \alpha_k u_k + 0u_{k+1} + \ldots + 0u_n = 0$$

and $S$ is also linearly dependent.
Proof

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and \( S \) is also linearly dependent.

2. Follows from (1) by contraposition.
(cont.)

3. “⇒”: Assume $S_{\text{ext}}$ is linearly independent. Then $\mathbf{v}$ can’t be a linear combination of $\mathbf{u}_1, \ldots, \mathbf{u}_n$. 
(cont.)

3. “$\implies$”: Assume $S_{\text{ext}}$ is linearly independent. Then $v$ can’t be a linear combination of $u_1, \ldots, u_n$.

“$\Leftarrow$”: Assume $v \notin \text{span}(S)$ and consider

$$\alpha_1 u_1 + \alpha_2 u_2 + \ldots + \alpha_n u_n + \alpha_{n+1} v = 0.$$ 

First, $\alpha_{n+1} = 0$ since otherwise $v \in \text{span}(S)$. 
"\[\text{(cont.)}\]

\[\begin{align*}
&\implies:\text{ Assume } S_{\text{ext}} \text{ is linearly independent. Then } v \text{ can't be a linear combination of } u_1, \ldots, u_n. \\
&\iff:\text{ Assume } v \notin \text{span}(S) \text{ and consider} \\
&\quad \alpha_1 u_1 + \alpha_2 u_2 + \ldots + \alpha_n u_n + \alpha_{n+1} v = 0. \\
\end{align*}\]

First, \(\alpha_{n+1} = 0\) since otherwise \(v \in \text{span}(S)\).

That leaves
\[\alpha_1 u_1 + \alpha_2 u_2 + \ldots + \alpha_n u_n = 0.\]

However, the linear independence of \(S\) implies \(\alpha_i = 0\), \(i = 1, \ldots, n\), and therefore \(S_{\text{ext}}\) is linearly independent.
We know that the columns of an $m \times n$ matrix $A$ are linearly independent if and only if $\text{rank}(A) = n$.

Here $A = (u_1 \ u_2 \ \cdots \ u_n)$ with $u_i \in \mathbb{R}^m$.

If $n > m$, then $\text{rank}(A) \leq m$ and $S$ must be linearly dependent.
Outline

1. Spaces and Subspaces
2. Four Fundamental Subspaces
3. Linear Independence
4. Bases and Dimension
5. More About Rank
6. Classical Least Squares
7. Kriging as best linear unbiased predictor
Earlier we introduced the concept of a spanning set of a vector space \( V \), i.e.,

\[
V = \text{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}
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Bases and Dimension

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Now

Definition

Consider a vector space \( \mathcal{V} \) with spanning set \( S \). If \( S \) is also linearly independent then we call it a basis of \( \mathcal{V} \).
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Example

1. \( \{ \mathbf{e}_1, \ldots, \mathbf{e}_n \} \) is the standard basis for \( \mathbb{R}^n \).
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Example
1. \( \{\mathbf{e}_1, \ldots, \mathbf{e}_n\} \) is the standard basis for \( \mathbb{R}^n \).
2. The columns/rows of an \( n \times n \) matrix \( A \) with \( \text{rank}(A) = n \) form a basis for \( \mathbb{R}^n \).
Remark

*Linear algebra deals with* finite-dimensional *linear spaces.*
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- \textit{infinitely differentiable functions with Taylor (polynomial) basis}

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\{1, x, x^2, x^3, \ldots\}
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Remark

Linear algebra deals with finite-dimensional linear spaces.

Functional analysis can be considered as infinite-dimensional linear algebra, where the linear spaces are usually function spaces such as

- infinitely differentiable functions with Taylor (polynomial) basis
  \( \{1, x, x^2, x^3, \ldots\} \)

- square integrable functions with Fourier basis
  \( \{1, \sin(x), \cos(x), \sin(2x), \cos(2x), \ldots\} \)
Earlier we mentioned the idea of **minimal spanning sets**.
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**Theorem**

Let $\mathcal{V}$ be a subspace of $\mathbb{R}^m$ and let

$$\mathcal{B} = \{b_1, b_2, \ldots, b_n\} \subseteq \mathcal{V}.$$  

The following are equivalent:

1. $\mathcal{B}$ is a basis for $\mathcal{V}$.
2. $\mathcal{B}$ is a minimal spanning set for $\mathcal{V}$.
3. $\mathcal{B}$ is a maximal linearly independent subset of $\mathcal{V}$.
Earlier we mentioned the idea of minimal spanning sets.

**Theorem**

Let \( \mathcal{V} \) be a subspace of \( \mathbb{R}^m \) and let

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The following are equivalent:

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2. \( \mathcal{B} \) is a minimal spanning set for \( \mathcal{V} \).
3. \( \mathcal{B} \) is a maximal linearly independent subset of \( \mathcal{V} \).

**Remark**

We say “a basis” here since \( \mathcal{V} \) can have many different bases.
Proof

Since it is difficult to directly relate (2) and (3), our strategy will be to prove

- Show (1) \implies (2) and (2) \implies (1), so that (1) \iff (2).
- Show (1) \implies (3) and (3) \implies (1), so that (1) \iff (3).

Then — by transitivity — we will also have (2) \iff (3).
Proof (cont.)

(1) \implies (2): Assume \( B \) is a basis (i.e., a linearly independent spanning set) of \( V \) and show that it is minimal.
Proof (cont.)

(1) \(\Rightarrow\) (2): Assume \(B\) is a basis (i.e., a linearly independent spanning set) of \(V\) and show that it is minimal.

Assume \(B\) is not minimal, i.e., we can find a smaller spanning set \(\{x_1, \ldots, x_k\}\) for \(V\) with \(k \leq n\) elements.
Proof (cont.)

(1) $\implies$ (2): Assume $\mathcal{B}$ is a basis (i.e., a linearly independent spanning set) of $\mathcal{V}$ and show that it is minimal.

Assume $\mathcal{B}$ is not minimal, i.e., we can find a smaller spanning set $\{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$ for $\mathcal{V}$ with $k \leq n$ elements.

But then

$$b_j = \sum_{i=1}^{k} \alpha_{ij} x_i, \quad j = 1, \ldots, n,$$

or
Proof (cont.)

(1) $\implies$ (2): Assume $\mathcal{B}$ is a basis (i.e., a linearly independent spanning set) of $\mathcal{V}$ and show that it is minimal.

Assume $\mathcal{B}$ is not minimal, i.e., we can find a smaller spanning set $\{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$ for $\mathcal{V}$ with $k \leq n$ elements.

But then

$$
\mathbf{b}_j = \sum_{i=1}^{k} \alpha_{ij} \mathbf{x}_i, \quad j = 1, \ldots, n,
$$

or

$$
\mathcal{B} = \mathbf{X} \mathbf{A},
$$

where

$$
\mathcal{B} = \begin{pmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_n \end{pmatrix} \in \mathbb{R}^{m \times n}, \\
\mathbf{X} = \begin{pmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_k \end{pmatrix} \in \mathbb{R}^{m \times k}, \\
[A]_{ij} = \alpha_{ij}, \quad \mathbf{A} \in \mathbb{R}^{k \times n}.
$$
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Proof (cont.)

Now, \( \text{rank}(A) \leq k < n \), which implies \( N(A) \) is nontrivial, i.e., there exists a \( z \neq 0 \) such that

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Az = 0.
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Now, $\text{rank}(A) \leq k < n$, which implies $N(A)$ is nontrivial, i.e., there exists a $z \neq 0$ such that
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But then
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and therefore $N(B)$ is nontrivial.

However, since $B$ is a basis, the columns of $B$ are linearly independent (i.e., $N(B) = \{0\}$) — and that is a contradiction.
**Proof (cont.)**

Now, \( \text{rank}(A) \leq k < n \), which implies \( N(A) \) is nontrivial, i.e., there exists a \( z \neq 0 \) such that

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A z = 0.
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But then

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and therefore \( N(B) \) is nontrivial.

However, since \( B \) is a basis, the columns of \( B \) are linearly independent (i.e., \( N(B) = \{0\} \)) — and that is a *contradiction*.

Therefore, \( B \) has to be minimal.
Proof (cont.)

(2) $\implies$ (1): Assume $B$ is a minimal spanning set and show that it must also be linearly independent.
Proof (cont.)

(2) $\implies$ (1): Assume $\mathcal{B}$ is a minimal spanning set and show that it must also be linearly independent.

This is clear since

- if $\mathcal{B}$ were linearly dependent,
- then we would be able to remove at least one vector from $\mathcal{B}$ and still have a spanning set
- but then it would not have been minimal.
Proof (cont.)

(3) \implies (1): Assume \( B \) is a maximal linearly independent subset of \( V \) and show that \( B \) is a basis of \( V \).
Proof (cont.)

(3) \implies (1): Assume \( \mathcal{B} \) is a maximal linearly independent subset of \( \mathcal{V} \) and show that \( \mathcal{B} \) is a basis of \( \mathcal{V} \).

Assume that \( \mathcal{B} \) is not a basis, i.e., there exists a \( \mathbf{v} \in \mathcal{V} \) such that \( \mathbf{v} \not\in \text{span}\{\mathbf{b}_1, \ldots, \mathbf{b}_n\} \).
Proof (cont.)

(3) \implies (1): Assume \( B \) is a maximal linearly independent subset of \( \mathcal{V} \) and show that \( B \) is a basis of \( \mathcal{V} \).

Assume that \( B \) is not a basis, i.e., there exists a \( \mathbf{v} \in \mathcal{V} \) such that \( \mathbf{v} \notin \text{span}\{\mathbf{b}_1, \ldots, \mathbf{b}_n\} \).

Then — by an earlier theorem — the extension set \( B \cup \{\mathbf{v}\} \) is linearly independent.
Proof (cont.)

(3) $\implies$ (1): Assume $\mathcal{B}$ is a maximal linearly independent subset of $\mathcal{V}$ and show that $\mathcal{B}$ is a basis of $\mathcal{V}$.

Assume that $\mathcal{B}$ is not a basis, i.e., there exists a $\mathbf{v} \in \mathcal{V}$ such that $\mathbf{v} \not\in \text{span}\{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$.

Then — by an earlier theorem — the extension set $\mathcal{B} \cup \{\mathbf{v}\}$ is linearly independent.

But this contradicts the maximality of $\mathcal{B}$, so that $\mathcal{B}$ has to be a basis.
Proof (cont.)

(1) \implies (3): Assume \( \mathcal{B} \) is a basis, but not a maximal linearly independent subset of \( \mathcal{V} \), and show that this leads to a contradiction.
Proof (cont.)

(1) \implies (3): Assume \( B \) is a basis, but not a maximal linearly independent subset of \( \mathcal{V} \), and show that this leads to a contradiction.

Let

\[ \mathcal{Y} = \{ y_1, \ldots, y_k \} \subseteq \mathcal{V}, \quad \text{with } k > n \]

be a maximal linearly independent subset of \( \mathcal{V} \) (note that such a set always exists).
Proof (cont.)

(1) \implies (3): Assume \( \mathcal{B} \) is a basis, but not a maximal linearly independent subset of \( \mathcal{V} \), and show that this leads to a contradiction.

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But then \( \mathcal{Y} \) must be a basis for \( \mathcal{V} \) by our “(1) \implies (3)” argument.
Proof (cont.)

(1) $\implies$ (3): Assume $\mathcal{B}$ is a basis, but not a maximal linearly independent subset of $\mathcal{V}$, and show that this leads to a contradiction.

Let

$$\mathcal{Y} = \{y_1, \ldots, y_k\} \subseteq \mathcal{V}, \quad \text{with } k > n$$

be a maximal linearly independent subset of $\mathcal{V}$ (note that such a set always exists).

But then $\mathcal{Y}$ must be a basis for $\mathcal{V}$ by our “(1) $\implies$ (3)” argument.

On the other hand, $\mathcal{Y}$ has more vectors than $\mathcal{B}$ and a basis has to be a minimal spanning set.

Therefore $\mathcal{B}$ has to already be a maximal linearly independent subset of $\mathcal{V}$. $\square$
Remark

Above we remarked that $B$ is not unique, i.e., a vector space $V$ can have many different bases.
Remark
Above we remarked that $\mathcal{B}$ is not unique, i.e., a vector space $\mathcal{V}$ can have many different bases.

However, the number of elements in all of these bases is unique.

Definition
The dimension of the vector space $\mathcal{V}$ is given by

$$\dim \mathcal{V} = \text{the number of elements in any basis of } \mathcal{V}.$$ 

Special case: by convention

$$\dim \{0\} = 0.$$
Example

Consider

\[ \mathcal{P} = \text{span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\} \subset \mathbb{R}^3. \]
Example

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\[ \mathcal{P} = \text{span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\} \subset \mathbb{R}^3. \]

Geometrically, \( \mathcal{P} \) corresponds to the plane \( z = 0 \), i.e., the \( xy \)-plane.
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Geometrically, \( \mathcal{P} \) corresponds to the plane \( z = 0 \), i.e., the \( xy \)-plane.

Note that \( \dim \mathcal{P} = 2 \).
Example

Consider

\[ P = \text{span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\} \subset \mathbb{R}^3. \]

Geometrically, \( P \) corresponds to the plane \( z = 0 \), i.e., the \( xy \)-plane.

Note that \( \dim P = 2 \).

Moreover, any subspace of \( \mathbb{R}^3 \) has dimension at most 3.
In general,

**Theorem**

Let $\mathcal{M}$ and $\mathcal{N}$ be vector spaces such that $\mathcal{M} \subseteq \mathcal{N}$. Then

1. $\dim \mathcal{M} \leq \dim \mathcal{N}$,
2. $\dim \mathcal{M} = \dim \mathcal{N} \implies \mathcal{M} = \mathcal{N}$.
In general,

**Theorem**

Let $\mathcal{M}$ and $\mathcal{N}$ be vector spaces such that $\mathcal{M} \subseteq \mathcal{N}$. Then

1. $\dim \mathcal{M} \leq \dim \mathcal{N}$,
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**Proof.**

See [Mey00].
Back to the 4 fundamental subspaces

Consider an $m \times n$ matrix $A$ with $\text{rank}(A) = r$. 

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$R(A)$ We know that

$$R(A) = \text{span}\{\text{columns of } A\}.$$

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Consider an $m \times n$ matrix $A$ with rank($A$) = $r$.

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$$R(A) = \text{span}\{\text{columns of } A\}.$$  

If rank($A$) = $r$, then only $r$ columns of $A$ are linearly independent, i.e.,

$$\dim R(A) = r.$$
Consider an $m \times n$ matrix $A$ with $\text{rank}(A) = r$.

$\mathcal{R}(A)$ We know that

$$\mathcal{R}(A) = \text{span}\{\text{columns of } A\}.$$

If $\text{rank}(A) = r$, then only $r$ columns of $A$ are linearly independent, i.e.,

$$\dim \mathcal{R}(A) = r.$$

A basis of $\mathcal{R}(A)$ is given by the basic columns of $A$ (determined via a row echelon form $U$).
$R(A^T)$  We know that

$$R(A^T) = \text{span}\{\text{rows of } A\}.$$
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\[ R(A^T) = \text{span}\{\text{rows of } A\}. \]

Again, \( \text{rank}(A) = r \) implies that only \( r \) rows of \( A \) are linearly independent, i.e.,

\[ \text{dim } R(A^T) = r. \]
We know that 

\[ R(A^T) = \text{span}\{\text{rows of } A\} \].

Again, \( \text{rank}(A) = r \) implies that only \( r \) rows of \( A \) are linearly independent, i.e.,

\[ \text{dim} \ R(A^T) = r. \]

A basis of \( R(A^T) \) is given by the nonzero rows of \( U \) (from the LU factorization of \( A \)).
One of our earlier theorems states that the last \( m - r \) rows of \( P \) span \( N(A^T) \) (where \( P \) is nonsingular such that \( PA = U \) is in row echelon form).
$N(A^T)$ One of our earlier theorems states that the last $m - r$ rows of $P$ span $N(A^T)$ (where $P$ is nonsingular such that $PA = U$ is in row echelon form).

Since $P$ is nonsingular these rows are linearly independent and so

$$\dim N(A^T) = m - r.$$
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\[ \text{dim } N(A^T) = m - r. \] 

\[ \text{A basis of } N(A^T) \text{ is given by the last } m - r \text{ rows of } P. \]
$N(A)$ Replace $A$ by $A^T$ above so that

$$\dim N\left(\left(A^T\right)^T\right) = n - \text{rank}(A^T) = n - r$$

so that

$$\dim N(A) = n - r.$$
Replace $A$ by $A^T$ above so that

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so that

$$\dim N(A) = n - r.$$

A basis of $N(A)$ is given by the $n - r$ linearly independent solutions of $Ax = 0$. 
Theorem

For any $m \times n$ matrix $A$ we have

$$\dim \mathcal{R}(A) + \dim \mathcal{N}(A) = n.$$
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For any $m \times n$ matrix $A$ we have

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$$\dim R(A) + \dim N(A) = n.$$  

This follows directly from the above discussion of $R(A)$ and $N(A)$. 

The theorem shows that there is always a balance between the rank of $A$ and the dimension of its nullspace.
Example

Find the dimension and a basis for

\[ S = \text{span} \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 6 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 6 \\ 4 \end{pmatrix}, \begin{pmatrix} 3 \\ 6 \\ 9 \\ 5 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 6 \\ 3 \end{pmatrix} \right\}. \]
Example

Find the dimension and a basis for

\[ S = \text{span} \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 6 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 6 \\ 4 \end{pmatrix}, \begin{pmatrix} 3 \\ 6 \\ 9 \\ 6 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 6 \\ 3 \end{pmatrix} \right\}. \]

Before we even do any calculations we know that

\[ S \subseteq \mathbb{R}^4, \quad \text{so that } \dim S \leq 4. \]
Example

Find the dimension and a basis for

\[ \mathcal{S} = \text{span} \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 6 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 6 \\ 4 \end{pmatrix}, \begin{pmatrix} 3 \\ 6 \\ 6 \\ 9 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 6 \\ 3 \end{pmatrix} \right\} . \]

Before we even do any calculations we know that

\[ \mathcal{S} \subseteq \mathbb{R}^4, \quad \text{so that } \dim \mathcal{S} \leq 4. \]

We will now answer this question in two different ways using

\[ A = \begin{pmatrix} 1 & 2 & 2 & 3 & 1 \\ 2 & 4 & 4 & 6 & 2 \\ 3 & 6 & 6 & 9 & 6 \\ 1 & 2 & 4 & 5 & 3 \end{pmatrix}. \]
Example (cont.)

Via $R(A)$, i.e., by finding the basic columns of $A$:

\[
A = \begin{pmatrix}
1 & 2 & 2 & 3 & 1 \\
2 & 4 & 4 & 6 & 2 \\
3 & 6 & 6 & 9 & 6 \\
1 & 2 & 4 & 5 & 3 \\
\end{pmatrix}
\]

\[\rightarrow \quad \text{G.-J.} \quad E_A = \begin{pmatrix}
1 & 2 & 0 & 1 & 0 \\
0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

Therefore, $\dim S = 3$ and $S = \text{span}\begin{pmatrix}1 \\ 2 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix}2 \\ 4 \\ 6 \\ 4 \end{pmatrix}, \begin{pmatrix}1 \\ 2 \\ 6 \\ 3 \end{pmatrix}$ since the basic columns of $E_A$ are the first, third and fifth columns.
Example (cont.)

Via \( R(A) \), i.e., by finding the basic columns of \( A \):

\[
A = \begin{pmatrix}
1 & 2 & 2 & 3 & 1 \\
2 & 4 & 4 & 6 & 2 \\
3 & 6 & 6 & 9 & 6 \\
1 & 2 & 4 & 5 & 3
\end{pmatrix}
\xrightarrow{G.-J.}
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1 & 2 & 0 & 1 & 0 \\
0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

Therefore, \( \text{dim} \ S = 3 \) and

\[
S = \text{span} \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 6 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 6 \\ 3 \\ 2 \end{pmatrix} \right\}
\]

since the basic columns of \( E_A \) are the first, third and fifth columns.
Example (cont.)

Via $R(A^T)$, i.e., $R(A) = \text{span}\{\text{rows of } A^T\}$, i.e., we need the nonzero rows of $U$ (from the LU factorization of $A^T$):

\[
A^T = \begin{pmatrix}
1 & 2 & 3 & 1 \\
2 & 4 & 6 & 2 \\
2 & 4 & 6 & 4 \\
3 & 6 & 9 & 4 \\
1 & 2 & 6 & 3
\end{pmatrix}
\]

zero out $[A^T]_{*,1}$

\[
\begin{pmatrix}
1 & 2 & 3 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 \\
0 & 0 & 0 & 2 \\
0 & 0 & 3 & 2
\end{pmatrix}
\]

permute

\[
\begin{pmatrix}
1 & 2 & 3 & 1 \\
0 & 0 & 3 & 2 \\
0 & 0 & 0 & 2 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
= U
\]
Example (cont.)

Via $R(A^T)$, i.e., $R(A) = \text{span}\{\text{rows of } A^T\}$, i.e., we need the nonzero rows of $U$ (from the LU factorization of $A^T$):

$$A^T = \begin{pmatrix} 1 & 2 & 3 & 1 \\ 2 & 4 & 6 & 2 \\ 2 & 4 & 6 & 4 \\ 3 & 6 & 9 & 4 \\ 1 & 2 & 6 & 3 \end{pmatrix}$$

zero out $[A^T]_{*,1}$

Therefore, $\text{dim } S = 3$ and

$$S = \text{span} \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

since the nonzero rows of $U$ are the first, second and third rows.
Example

Extend

\[ S = \text{span} \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 6 \\ 3 \end{pmatrix} \right\} \]

to a basis for \( \mathbb{R}^4 \).
Example

Extend

$$S = \text{span} \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 6 \\ 3 \end{pmatrix} \right\}$$

to a basis for $$\mathbb{R}^4$$.

The procedure will be to augment the columns of $$S$$ by an identity matrix, i.e., to form

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 2 & 2 & 0 & 1 & 0 & 0 \\ 3 & 6 & 0 & 0 & 1 & 0 \\ 1 & 3 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and then to get a basis via the basic columns of $$U$$. 
Example (cont.)

\[ A = \begin{pmatrix}
1 & 1 & 1 & 0 & 0 & 0 \\
2 & 2 & 0 & 1 & 0 & 0 \\
3 & 6 & 0 & 0 & 1 & 0 \\
1 & 3 & 0 & 0 & 0 & 1 \\
\end{pmatrix} \rightarrow \]

so that the basic columns are \([A]_1, [A]_2, [A]_3, [A]_4\) and

\[ R_4 = \text{span} \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 6 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}. \]
Example (cont.)

\[ A = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 2 & 2 & 0 & 1 & 0 & 0 \\ 3 & 6 & 0 & 0 & 1 & 0 \\ 1 & 3 & 0 & 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -2 & 1 & 0 & 0 \\ 0 & 3 & -3 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{pmatrix} \]

so that the basic columns are \( A_1 \), \( A_2 \), \( A_3 \), and \( A_4 \) and

\[ R_4 = \text{span} \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 6 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\} \]
Example (cont.)

\[ A = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 2 & 2 & 0 & 1 & 0 & 0 \\ 3 & 6 & 0 & 0 & 1 & 0 \\ 1 & 3 & 0 & 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -2 & 1 & 0 & 0 \\ 0 & 3 & -3 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{pmatrix} \]

\[ \rightarrow \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \\ 0 & 0 & -\frac{3}{2} & 0 & 1 & -\frac{3}{2} \\ 0 & 0 & -2 & 1 & 0 & 0 \end{pmatrix} \]

so that the basic columns are \( [A]_1^*, [A]_2^*, [A]_3^*, [A]_4^* \) and 

\[ R_4 = \text{span} \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 6 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -\frac{3}{2} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -2 \\ 1 \end{pmatrix} \right\} \]
Example (cont.)

\[ A = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 2 & 2 & 0 & 1 & 0 & 0 \\ 3 & 6 & 0 & 0 & 1 & 0 \\ 1 & 3 & 0 & 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -2 & 1 & 0 & 0 \\ 0 & 3 & -3 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{pmatrix} \]

\[ \rightarrow \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \\ 0 & 0 & -\frac{3}{2} & 0 & 1 & -\frac{3}{2} \\ 0 & 0 & -2 & 1 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \\ 0 & 0 & -\frac{3}{2} & 0 & 1 & -\frac{3}{2} \\ 0 & 0 & 0 & 1 & -\frac{4}{3} & 2 \end{pmatrix} \]
Bases and Dimension

Example (cont.)

\[ A = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 2 & 2 & 0 & 1 & 0 & 0 \\ 3 & 6 & 0 & 0 & 1 & 0 \\ 1 & 3 & 0 & 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 3 & 3 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{pmatrix} \]

\[ \rightarrow \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \\ 0 & 0 & 3 & 7 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \\ 0 & 0 & 3 & 7 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix} \]

so that the basic columns are \([A]_1, [A]_2, [A]_3, [A]_4\) and

\[ \mathbb{R}^4 = \text{span} \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 6 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \end{pmatrix}, \begin{pmatrix} 0 \end{pmatrix}, \begin{pmatrix} 0 \end{pmatrix}, \begin{pmatrix} 0 \end{pmatrix}, \begin{pmatrix} 1 \end{pmatrix} \right\}. \]
Earlier we defined the **sum of subspaces** $\mathcal{X}$ and $\mathcal{Y}$ as

$$\mathcal{X} + \mathcal{Y} = \{ x + y : x \in \mathcal{X}, \ y \in \mathcal{Y} \}$$
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**Theorem**

*If $\mathcal{X}, \mathcal{Y}$ are subspaces of $\mathcal{V}$, then*

$$\dim(\mathcal{X} + \mathcal{Y}) = \dim \mathcal{X} + \dim \mathcal{Y} - \dim(\mathcal{X} \cap \mathcal{Y}).$$
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**Proof.**

See [Mey00], but the basic idea is pretty clear. We want to avoid double counting.
Corollary

Let $A$ and $B$ be $m \times n$ matrices. Then

$$\text{rank}(A + B) \leq \text{rank}(A) + \text{rank}(B).$$
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$$\text{rank}(A + B) \leq \text{rank}(A) + \text{rank}(B).$$

Proof

First we note that

$$R(A + B) \subseteq R(A) + R(B)$$  \hspace{1cm} (4)
Corollary

Let $A$ and $B$ be $m \times n$ matrices. Then

$$\text{rank}(A + B) \leq \text{rank}(A) + \text{rank}(B).$$

Proof

First we note that

$$R(A + B) \subseteq R(A) + R(B) \quad (4)$$

since for any $b \in R(A + B)$ we have

$$b = (A + B)x = Ax + Bx \in R(A) + R(B).$$
(cont.)

Now,

\[ \text{rank}(A + B) = \dim R(A + B) \]
Now,

\[
\text{rank}(A + B) = \dim R(A + B)
\]

\[(4)\]

\[
\leq \dim (R(A) + R(B))
\]
(cont.)

Now,

$$\text{rank}(A + B) = \dim R(A + B)$$

$$\leq \dim (R(A) + R(B))$$

$$\overset{\text{Thm}}{=} \dim R(A) + \dim R(B) - \dim (R(A) \cap R(B))$$
(cont.)

Now,

\[
\text{rank}(A + B) = \dim R(A + B)
\]

\[(4) \quad \leq \dim (R(A) + R(B))\]

Thm \quad \equiv \quad \dim R(A) + \dim R(B) - \dim (R(A) \cap R(B))

\leq \dim R(A) + \dim R(B)

= \text{rank}(A) + \text{rank}(B)
Outline

1. Spaces and Subspaces
2. Four Fundamental Subspaces
3. Linear Independence
4. Bases and Dimension
5. More About Rank
6. Classical Least Squares
7. Kriging as best linear unbiased predictor
More About Rank

We know that $A \sim B$ if and only if $\text{rank}(A) = \text{rank}(B)$. 
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Thus (for invertible $P, Q$), $PAQ = B$ implies $\text{rank}(A) = \text{rank}(PAQ)$. 
More About Rank

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Thus (for invertible $P$, $Q$), $PAQ = B$ implies $\text{rank}(A) = \text{rank}(PAQ)$.

As we now show, it is a general fact that multiplication by a nonsingular matrix does not change the rank of a given matrix.
More About Rank

We know that $A \sim B$ if and only if $\text{rank}(A) = \text{rank}(B)$.

Thus (for invertible $P, Q$), $PAQ = B$ implies $\text{rank}(A) = \text{rank}(PAQ)$.

As we now show, it is a general fact that multiplication by a nonsingular matrix does not change the rank of a given matrix.

Moreover, multiplication by an arbitrary matrix can only lower the rank.

Theorem

Let $A$ be an $m \times n$ matrix, and let $B$ by $n \times p$. Then

$$\text{rank}(AB) = \text{rank}(B) - \dim (N(A) \cap R(B)).$$

Remark

Note that if $A$ is nonsingular, then $N(A) = \{0\}$ so that $\dim (N(A) \cap R(B)) = 0$ and $\text{rank}(AB) = \text{rank}(B)$. 
Proof

Let \( S = \{x_1, x_2, \ldots, x_s\} \) be a basis for \( N(A) \cap R(B) \).
Proof
Let $\mathcal{S} = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_s\}$ be a basis for $N(A) \cap R(B)$.

Since $N(A) \cap R(B) \subseteq R(B)$ we know that

$$\dim(R(B)) = s + t, \quad \text{for some } t \geq 0.$$
Proof
Let $\mathcal{S} = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_s\}$ be a basis for $N(A) \cap R(B)$.

Since $N(A) \cap R(B) \subseteq R(B)$ we know that

$$\dim(R(B)) = s + t, \quad \text{for some } t \geq 0.$$ 

We can construct an extension set such that

$$\mathcal{B} = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_s, \mathbf{z}_1, \ldots, \mathbf{z}_2, \ldots, \mathbf{z}_t\}$$

is a basis for $R(B)$. 

(cont.)

If we can show that \( \dim(R(AB)) = t \) then

\[
\text{rank}(B) = \dim(R(B)) = s + t = \dim(N(A) \cap R(B)) + \dim(R(AB)),
\]

and we are done.
(cont.)

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and we are done.

Therefore, we now show that \( \dim(R(AB)) = t \).

In particular, we show that

\[
\mathcal{T} = \{Az_1, Az_2, \ldots, Az_t\}
\]

is a basis for \( R(AB) \).
(cont.)

If we can show that \( \dim(R(AB)) = t \) then

\[
\text{rank}(B) = \dim(R(B)) = s + t = \dim(N(A) \cap R(B)) + \dim(R(AB)),
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and we are done.

Therefore, we now show that \( \dim(R(AB)) = t \).

In particular, we show that

\[
\mathcal{T} = \{Az_1, Az_2, \ldots, Az_t\}
\]

is a basis for \( R(AB) \).

We do this by showing that

1. \( \mathcal{T} \) is a spanning set for \( R(AB) \),
2. \( \mathcal{T} \) is linearly independent.
Spanning set: Consider an arbitrary \( b \in R(AB) \). It can be written as

\[
b = ABy
\]

for some \( y \).
Spanning set: Consider an arbitrary \( b \in R(AB) \). It can be written as

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b = ABy \quad \text{for some } y.
\]

But then \( By \in R(B) \), so that

\[
By = \sum_{i=1}^{s} \xi_i x_i + \sum_{j=1}^{t} \eta_j z_j
\]

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\]
(cont.)

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and

\[
b = ABy = \sum_{i=1}^{s} \xi_i Ax_i + \sum_{j=1}^{t} \eta_j Az_j = \sum_{j=1}^{t} \eta_j Az_j
\]

since \( x_i \in N(A) \).
Linear independence: Let’s use the definition of linear independence and look at

\[ \sum_{i=1}^{t} \alpha_i A z_i = 0 \iff A \sum_{i=1}^{t} \alpha_i z_i = 0. \]
(cont.)

**Linear independence:** Let’s use the definition of linear independence and look at

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But we also have \( z_i \in B \), i.e., \( \sum_{i=1}^{t} \alpha_i z_i \in R(B) \).
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And so together

\[ \sum_{i=1}^{t} \alpha_i z_i \in N(A) \cap R(B). \]
(cont.)

Now, since \( S = \{x_1, \ldots, x_s\} \) is a basis for \( N(A) \cap R(B) \) we have

\[
\sum_{i=1}^{t} \alpha_i z_i = \sum_{j=1}^{s} \beta_j x_j \quad \iff
\]

\[
\sum_{i=1}^{t} \alpha_i z_i = \sum_{j=1}^{s} \beta_j x_j = 0.
\]

But \( B = \{x_1, \ldots, x_s, z_1, \ldots, z_t\} \) is linearly independent, so that

\[
\alpha_1 = \cdots = \alpha_t = \beta_1 = \cdots = \beta_s = 0
\]

and therefore \( T \) is also linearly independent. \( \square \)
Now, since $S = \{x_1, \ldots, x_s\}$ is a basis for $N(A) \cap R(B)$ we have

$$\sum_{i=1}^{t} \alpha_i z_i = \sum_{j=1}^{s} \beta_j x_j \iff \sum_{i=1}^{t} \alpha_i z_i - \sum_{j=1}^{s} \beta_j x_j = 0.$$
(cont.)

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\( \alpha_1 = \cdots = \alpha_t = \beta_1 = \cdots = \beta_s = 0 \) and
More About Rank

(cont.)

Now, since \( S = \{ x_1, \ldots, x_s \} \) is a basis for \( N(A) \cap R(B) \) we have

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\]

But \( B = \{ x_1, \ldots, x_s, z_1, \ldots, z_t \} \) is linearly independent, so that

\[
\alpha_1 = \cdots = \alpha_t = \beta_1 = \cdots = \beta_s = 0 \quad \text{and therefore} \quad T \quad \text{is also linearly independent.} \quad \square
\]
It turns out that \( \dim(N(A) \cap R(B)) \) is relatively difficult to determine.
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Therefore, the following upper and lower bounds for $\text{rank}(AB)$ are useful.
It turns out that \( \text{dim}(N(A) \cap R(B)) \) is relatively difficult to determine. Therefore, the following upper and lower bounds for \( \text{rank}(AB) \) are useful.

**Theorem**

Let \( A \) be an \( m \times n \) matrix, and let \( B \) by \( n \times p \). Then

1. \( \text{rank}(AB) \leq \min\{\text{rank}(A), \text{rank}(B)\} \),
2. \( \text{rank}(AB) \geq \text{rank}(A) + \text{rank}(B) - n \).
Proof of (1)

We show that $\text{rank}(AB) \leq \text{rank}(A)$ \textbf{and} $\text{rank}(AB) \leq \text{rank}(B)$. 
Proof of (1)
We show that \( \text{rank}(AB) \leq \text{rank}(A) \) and \( \text{rank}(AB) \leq \text{rank}(B) \).

The previous theorem states

\[
\text{rank}(AB) = \text{rank}(B) - \dim(N(A) \cap R(B))
\]

To make things as tight as possible we take the smaller of the two upper bounds.
Proof of (1)

We show that \( \text{rank}(AB) \leq \text{rank}(A) \) and \( \text{rank}(AB) \leq \text{rank}(B) \).

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\[
\text{rank}(AB) = \text{rank}(B) - \dim(N(A) \cap R(B)) \leq \text{rank}(B).
\]

Similarly, \( \text{rank}(AB) = \text{rank}(AB)^T = \text{rank}(B^T A^T) \) as above \( \leq \text{rank}(A^T) = \text{rank}(A) \).

To make things as tight as possible we take the smaller of the two upper bounds.
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\[
\text{rank}(AB) = \text{rank}(B) - \dim(N(A) \cap R(B)) \geq 0 \leq \text{rank}(B).
\]

Similarly,

\[
\text{rank}(AB) = \text{rank}(AB)^T = \text{rank}(B^T A^T) \quad \text{as above} \leq \text{rank}(A^T) =
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\text{rank}(AB) = \text{rank}(B) - \dim(N(A) \cap R(B)) \leq \text{rank}(B).
\]

Similarly,

\[
\text{rank}(AB) = \text{rank}(AB)^T = \text{rank}(B^T A^T) \overset{\text{as above}}{\leq} \text{rank}(A^T) = \text{rank}(A).
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The previous theorem states

\[
\text{rank}(AB) = \text{rank}(B) - \dim(\mathcal{N}(A) \cap \mathcal{R}(B)) \leq \text{rank}(B).
\]

Similarly,

\[
\text{rank}(AB) = \text{rank}(AB)^T = \text{rank}(B^T A^T) \quad \text{as above} \quad \leq \text{rank}(A^T) = \text{rank}(A).
\]

To make things as tight as possible we take the smaller of the two upper bounds.
Proof of (2)

We begin by noting that $N(A) \cap R(B) \subseteq N(A)$.
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Proof of (2)

We begin by noting that \( N(A) \cap R(B) \subseteq N(A) \).

Therefore,

\[
dim(N(A) \cap R(B)) \leq dim(N(A)) = n - \text{rank}(A).
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Proof of (2)

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Therefore,

\[
\dim(N(A) \cap R(B)) \leq \dim(N(A)) = n - \text{rank}(A).
\]

But then (using the previous theorem)

\[
\text{rank}(AB) = \text{rank}(B) - \dim(N(A) \cap R(B))
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Proof of (2)

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Therefore,

$$\dim(N(A) \cap R(B)) \leq \dim(N(A)) = n - \text{rank}(A).$$

But then (using the previous theorem)

$$\text{rank}(AB) = \text{rank}(B) - \dim(N(A) \cap R(B))$$

$$\geq \text{rank}(B) - n + \text{rank}(A).$$
To prepare for our study of *least squares solutions*, where the matrices $A^TA$ and $AA^T$ are important, we prove
To prepare for our study of least squares solutions, where the matrices $A^T A$ and $AA^T$ are important, we prove

**Lemma**

*Let $A$ be a real $m \times n$ matrix. Then*

1. $\text{rank}(A^T A) = \text{rank}(AA^T) = \text{rank}(A)$.
2. $R(A^T A) = R(A^T)$, $R(AA^T) = R(A)$.
Proof

From our earlier theorem we know

$$\text{rank}(A^T A) = \text{rank}(A) - \dim(N(A^T) \cap R(A)).$$
Proof
From our earlier theorem we know

\[ \text{rank}(A^T A) = \text{rank}(A) - \dim(N(A^T) \cap R(A)). \]

For (1) to be true we need to show \( \dim(N(A^T) \cap R(A)) = 0 \), i.e.,
\( N(A^T) \cap R(A) = \{0\} \).
Proof

From our earlier theorem we know

\[ \text{rank}(A^T A) = \text{rank}(A) - \text{dim}(N(A^T) \cap R(A)). \]

For (1) to be true we need to show \( \text{dim}(N(A^T) \cap R(A)) = 0 \), i.e., \( N(A^T) \cap R(A) = \{0\} \).

This is true since

\[ x \in N(A^T) \cap R(A) \implies A^T x = 0 \text{ and } x = Ay \text{ for some } y. \]
Proof

From our earlier theorem we know

$$\text{rank}(A^T A) = \text{rank}(A) - \dim(N(A^T) \cap R(A)).$$

For (1) to be true we need to show $$\dim(N(A^T) \cap R(A)) = 0$$, i.e., $$N(A^T) \cap R(A) = \{0\}$$.

This is true since

$$x \in N(A^T) \cap R(A) \implies A^T x = 0 \text{ and } x = Ay \text{ for some } y.$$ 

Therefore (using $$x^T = y^T A^T$$)

$$x^T x = y^T A^T x$$
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From our earlier theorem we know

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Therefore (using \( x^T = y^T A^T \))

\[ x^T x = y^T A^T x = 0. \]
Proof

From our earlier theorem we know

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For (1) to be true we need to show \( \text{dim}(N(A^T) \cap R(A)) = 0 \), i.e., \( N(A^T) \cap R(A) = \{0\} \).
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Therefore (using \( x^T = y^T A^T \))

\[ x^T x = y^T A^T x = 0. \]

But

\[ x^T x = 0 \quad \iff \quad \sum_{i=1}^{m} x_i^2 = 0 \quad \implies \]
Proof

From our earlier theorem we know

\[ \text{rank}(A^T A) = \text{rank}(A) - \dim(N(A^T) \cap R(A)). \]

For (1) to be true we need to show \( \dim(N(A^T) \cap R(A)) = 0 \), i.e., \( N(A^T) \cap R(A) = \{0\} \).

This is true since

\[ x \in N(A^T) \cap R(A) \implies A^T x = 0 \text{ and } x = Ay \text{ for some } y. \]

Therefore (using \( x^T = y^T A^T \))

\[ x^T x = y^T A^T x = 0. \]

But

\[ x^T x = 0 \iff \sum_{i=1}^{m} x_i^2 = 0 \implies x = 0. \]
(cont.)

\[ \text{rank}(AA^T) = \text{rank}(A^T) \text{ obtained by switching } A \text{ and } A^T, \text{ and then use } \text{rank}(A^T) = \text{rank}(A). \]
(cont.)

\[ \text{rank}(AA^T) = \text{rank}(A^T) \] obtained by switching \( A \) and \( A^T \), and then use \( \text{rank}(A^T) = \text{rank}(A) \).

The first part of (2) follows from \( R(A^T A) \subseteq R(A^T) \) (see HW) and

\[ \text{dim}(R(A^T A)) = \text{rank}(A^T A) \]
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\[
\dim(R(A^T A)) = \text{rank}(A^T A) \overset{(1)}{=} \text{rank}(A^T) = \dim(R(A^T))
\]
(cont.)

\[ \text{rank}(AA^T) = \text{rank}(A^T) \]

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since for \( \mathcal{M} \subseteq \mathcal{N} \) with \( \dim \mathcal{M} = \dim \mathcal{N} \) one has \( \mathcal{M} = \mathcal{N} \) (from an earlier theorem).
(cont.)

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The other part of (2) follows by switching \( A \) and \( A^T \).
More About Rank

The first part of (3) follows from $N(A) \subseteq N(A^T A)$ (see HW) and

$$\dim(N(A)) = n - \text{rank}(A)$$
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using the same reasoning as above.
(cont.)
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The other part of (3) follows by switching $A$ and $A^T$. □
Connection to least squares and normal equations

Consider a — possibly inconsistent — linear system

\[ Ax = b \]

with \( m \times n \) matrix \( A \) (and \( b \notin R(A) \) if inconsistent).
Connection to least squares and normal equations

Consider a — possibly inconsistent — linear system

\[ Ax = b \]

with \( m \times n \) matrix \( A \) (and \( b \notin R(A) \) if inconsistent).

To find a “solution” we multiply both sides by \( A^T \) to get the normal equations:

\[ A^T Ax = A^T b, \]

where \( A^T A \) is an \( n \times n \) matrix.
Theorem

Let $A$ be an $m \times n$ matrix, $b$ an $m$-vector, and consider the normal equations

$$A^T Ax = A^T b$$

associated with $Ax = b$.

1. The normal equations are always consistent, i.e., for every $A$ and $b$ there exists at least one $x$ such that $A^T Ax = A^T b$.

2. If $Ax = b$ is consistent, then $A^T Ax = A^T b$ has the same solution set (the least squares solution of $Ax = b$).

3. $A^T Ax = A^T b$ has a unique solution if and only if $\text{rank}(A) = n$. Then

$$x = (A^T A)^{-1} A^T b,$$

regardless of whether $Ax = b$ is consistent or not.

4. If $Ax = b$ is consistent and has a unique solution, then the same holds for $A^T Ax = A^T b$ and $x = (A^T A)^{-1} A^T b$. 
Proof

(1) follows from our previous lemma, i.e.,

\[ A^T b \in R(A^T) = R(A^T A). \]
Proof

(1) follows from our previous lemma, i.e.,

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To show (2) we assume the \( p \) is some particular solution of \( Ax = b \), i.e., \( Ap = b \).
Proof
(1) follows from our previous lemma, i.e.,

\[ A^T b \in R(A^T) = R(A^T A). \]

To show (2) we assume the \( p \) is some particular solution of \( Ax = b \), i.e., \( Ap = b \).

If we multiply by \( A^T \), then

\[ A^T Ap = A^T p, \]

so that \( p \) is also a solution of the normal equations.
(cont.)

Now, the **general solution of** $Ax = b$ **is from the set** (see Problem 2 on HW#4)

$$S = p + N(A).$$
(cont.)
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Moreover, the general solution of $A^T A x = A^T b$ is of the form

$$p + N(A^T A)$$
Now, the general solution of $Ax = b$ is from the set (see Problem 2 on HW#4)

$$S = p + N(A).$$

Moreover, the general solution of $A^TAx = A^Tb$ is of the form

$$p + N(A^T A) \text{ lemma } p + N(A) = S.$$
(cont.)

For (3) we want to show that $A^T A x = A^T b$ has a unique solution if and only if $\text{rank}(A) = n$. 
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Since we showed earlier that $\text{rank}(A^T A) = \text{rank}(A)$ this part is done.

□
(cont.)

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Now, if $\text{rank}(A^T A) = n$ we know that $A^T A$ is invertible (even though $A^T$ and $A$ may not be) and therefore

$$A^T A x = A^T b \iff x = (A^T A)^{-1} A^T b.$$
(cont.)

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What we know immediately is that \( A^T A x = A^T b \) has a unique solution if and only if \( \text{rank}(A^T A) = n \).

Since we showed earlier that \( \text{rank}(A^T A) = \text{rank}(A) \) this part is done.

Now, if \( \text{rank}(A^T A) = n \) we know that \( A^T A \) is invertible (even though \( A^T \) and \( A \) may not be) and therefore

\[
A^T A x = A^T b \iff x = (A^T A)^{-1} A^T b.
\]

To show (4) we note that \( Ax = b \) has a unique solution if and only if \( \text{rank}(A) = n \). But \( \text{rank}(A^T A) = \text{rank}(A) \) and the rest follows from (3). \( \square \)
Remark

The normal equations are not recommended for serious computations since they are often rather ill-conditioned since one can show that

$$\text{cond}(A^T A) = \text{cond}(A)^2.$$ 

There's an example in [Mey00] that illustrates this fact.
Historical definition of rank

Let $A$ be an $m \times n$ matrix. Then $A$ has rank $r$ if there exists at least one nonsingular $r \times r$ submatrix of $A$ (and none larger).
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Example

The matrix

$$A = \begin{pmatrix}
1 & 2 & 2 & 3 & 1 \\
2 & 4 & 4 & 6 & 2 \\
3 & 6 & 6 & 9 & 6 \\
1 & 2 & 4 & 5 & 3
\end{pmatrix}$$

cannot have rank 4
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cannot have rank 4 since rows one and two are linearly dependent.
Historical definition of rank

Let \( A \) be an \( m \times n \) matrix. Then \( A \) has rank \( r \) if there exists at least one nonsingular \( r \times r \) submatrix of \( A \) (and none larger).

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\]

cannot have rank 4 since rows one and two are linearly dependent.

But \( \text{rank}(A) \geq 2 \)
Historical definition of rank

Let $A$ be an $m \times n$ matrix. Then $A$ has rank $r$ if there exists at least one nonsingular $r \times r$ submatrix of $A$ (and none larger).

Example

The matrix

\[
A = \begin{pmatrix}
1 & 2 & 2 & 3 & 1 \\
2 & 4 & 4 & 6 & 2 \\
3 & 6 & 6 & 9 & 6 \\
1 & 2 & 4 & 5 & 3
\end{pmatrix}
\]

cannot have rank 4 since rows one and two are linearly dependent.
But $\text{rank}(A) \geq 2$ since \begin{pmatrix}
9 & 6 \\
5 & 3
\end{pmatrix}
is nonsingular.
Example (cont.)

In fact, \( \text{rank}(A) = 3 \) since

\[
\begin{pmatrix}
4 & 6 & 2 \\
6 & 9 & 6 \\
4 & 5 & 3
\end{pmatrix}
\]

is nonsingular.
Example (cont.)

In fact, rank(A) = 3 since

$$\begin{pmatrix}
4 & 6 & 2 \\
6 & 9 & 6 \\
4 & 5 & 3
\end{pmatrix}$$

is nonsingular.

Note that other singular $3 \times 3$ submatrices are allowed, such as

$$\begin{pmatrix}
1 & 2 & 2 \\
2 & 4 & 4 \\
3 & 6 & 6
\end{pmatrix}.$$
Earlier we showed that

$$\text{rank}(AB) \leq \text{rank}(A),$$

i.e., multiplication by another matrix does not increase the rank of a given matrix, i.e., we can’t “fix” a singular system by multiplication.
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i.e., multiplication by another matrix does not increase the rank of a given matrix, i.e., we can’t “fix” a singular system by multiplication.

Now

**Theorem**

*Let A and E be \( m \times n \) matrices. Then*

\[ \text{rank}(A + E) \geq \text{rank}(A), \]

*provided the entries of E are “sufficiently small”.*
This theorem has at least two fundamental consequences of practical importance:
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- **Beware!!** A theoretically singular system may become nonsingular, i.e., have a “solution” — just due to round-off error.
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- **Beware!!** A theoretically singular system may become nonsingular, i.e., have a “solution” — just due to round-off error.

- We may want to intentionally “fix” a singular system, so that it has a “solution”. One such strategy is known as Tikhonov regularization, i.e.,

\[
Ax = b \quad \rightarrow \quad (A + \mu I)x = b,
\]

where \( \mu \) is a (small) regularization parameter.
Proof

We assume that \( \text{rank}(A) = r \) and that we have nonsingular \( P \) and \( Q \) such that we can convert \( A \) to rank normal form, i.e.,

\[
PAQ = \begin{pmatrix} I_r & O \\ O & O \end{pmatrix}.
\]
Proof

We assume that \( \text{rank}(A) = r \) and that we have nonsingular \( P \) and \( Q \) such that we can convert \( A \) to rank normal form, i.e.,

\[
PAQ = \begin{pmatrix}
I_r & 0 \\
0 & O
\end{pmatrix}
\]

Then — formally —

\[
PEQ = \begin{pmatrix}
E_{11} & E_{12} \\
E_{21} & E_{22}
\end{pmatrix}
\]

with appropriate blocks \( E_{ij} \).
More About Rank

Proof

We assume that \( \text{rank}(A) = r \) and that we have nonsingular \( P \) and \( Q \) such that we can convert \( A \) to rank normal form, i.e.,

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\]

Then — formally — \( PEQ = \begin{pmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{pmatrix} \) with appropriate blocks \( E_{ij} \).

This allows us to write

\[
P(A + E)Q = \begin{pmatrix} I_r + E_{11} & E_{12} \\ E_{21} & E_{22} \end{pmatrix}.
\]
(cont.)

Now, we note that

\[(I - B)(I + B + B^2 + \ldots + B^{k-1})\]
(cont.)

Now, we note that

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provided the entries of $B$ are “sufficiently small” (i.e., so that $B^k \rightarrow O$ for $k \rightarrow \infty$).
(cont.)
Now, we note that

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provided the entries of \(B\) are “sufficiently small” (i.e., so that \(B^k \rightarrow 0\) for \(k \rightarrow \infty\)).

Therefore \((I - B)^{-1}\) exists.
Now, we note that

$$(I - B)(I + B + B^2 + \ldots + B^{k-1}) = I - B^k \to I,$$

provided the entries of $B$ are “sufficiently small” (i.e., so that $B^k \to O$ for $k \to \infty$).

Therefore $(I - B)^{-1}$ exists.

This technique is known as the Neumann series expansion of the inverse of $I - B$. 

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MATH 532
(cont.)

Now, letting $B = -E_{11}$, we know that $(I_r + E_{11})^{-1}$ exists and we can write

$$
\begin{pmatrix}
I_r & O \\
-E_{21}(I_r + E_{11})^{-1} & I_{m-r}
\end{pmatrix}
\begin{pmatrix}
I_r + E_{11} & E_{12} \\
E_{21} & E_{22}
\end{pmatrix}
\begin{pmatrix}
I_r & -(I_r + E_{11})^{-1}E_{12} \\
O & I_{n-r}
\end{pmatrix}
= \begin{pmatrix}
I_r + E_{11} & O \\
O & S
\end{pmatrix},
$$

where $S = E_{22} - E_{21}(I_r + E_{11})^{-1}E_{12}$ is the Schur complement of $I + E_{11}$ in PAQ.
(cont.)
The Schur complement calculation shows that

\[ A + E \sim \begin{pmatrix} I_r + E_{11} & O \\ O & S \end{pmatrix}. \]
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But then this rank normal form with invertible diagonal blocks tells us

\[ \text{rank}(A + E) = \text{rank}(I_r + E_{11}) + \text{rank}(S) \]
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\[ A + E \sim \begin{pmatrix} I_r + E_{11} & O \\ O & S \end{pmatrix} \].

But then this rank normal form with invertible diagonal blocks tells us

\[
\text{rank}(A + E) = \text{rank}(I_r + E_{11}) + \text{rank}(S) \\
= \text{rank}(A) + \text{rank}(S) \\
\geq \text{rank}(A).
\]
Outline

1. Spaces and Subspaces
2. Four Fundamental Subspaces
3. Linear Independence
4. Bases and Dimension
5. More About Rank
6. Classical Least Squares
7. Kriging as best linear unbiased predictor
Linear least squares (linear regression)

Given: data \{ \((t_1, b_1), (t_2, b_2), \ldots, (t_m, b_m)\) \}

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<tr>
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</thead>
<tbody>
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Idea for best fit

Minimize the sum of the squares of the vertical distances of line from the data points.
Linear least squares (linear regression)

Given: data \{((t_1, b_1), (t_2, b_2), \ldots, (t_m, b_m))\}

Find: “best fit” by a line

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Minimize the sum of the squares of the vertical distances of line from the data points.
More precisely, let

$$f(t) = \alpha + \beta t$$

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\[
\sum_{i=1}^{m} (f(t_i) - b_i)^2 \rightarrow \min
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More precisely, let

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\[
\sum_{i=1}^{m} (f(t_i) - b_i)^2 = \sum_{i=1}^{m} (\alpha + \beta t_i - b_i)^2 \quad \rightarrow \quad \text{min}
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\[
\sum_{i=1}^{m} \varepsilon_i^2 = \sum_{i=1}^{m} (f(t_i) - b_i)^2
\]

\[
= \sum_{i=1}^{m} (\alpha + \beta t_i - b_i)^2 = G(\alpha, \beta) \rightarrow \text{min}
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From calculus, necessary (and sufficient) condition for minimum 

\[
\frac{\partial G(\alpha, \beta)}{\partial \alpha} = 0, \quad \frac{\partial G(\alpha, \beta)}{\partial \beta} = 0.
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From calculus, necessary (and sufficient) condition for minimum

\[
\frac{\partial G(\alpha, \beta)}{\partial \alpha} = 0, \quad \frac{\partial G(\alpha, \beta)}{\partial \beta} = 0.
\]

where

\[
\frac{\partial G(\alpha, \beta)}{\partial \alpha} = 2 \sum_{i=1}^{m} (\alpha + \beta t_i - b_i), \quad \frac{\partial G(\alpha, \beta)}{\partial \beta} = 2 \sum_{i=1}^{m} (\alpha + \beta t_i - b_i) t_i
\]
Equivalently,

\[
\begin{align*}
\left( \sum_{i=1}^{m} 1 \right) \alpha + \left( \sum_{i=1}^{m} t_i \right) \beta &= \sum_{i=1}^{m} b_i \\
\left( \sum_{i=1}^{m} t_i \right) \alpha + \left( \sum_{i=1}^{m} t_i^2 \right) \beta &= \sum_{i=1}^{m} b_i t_i
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which can be written as

\[
Qx = y
\]

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\end{align*}
\]

which can be written as

\[Qx = y\]

with

\[
Q = \begin{pmatrix}
\sum_{i=1}^{m} 1 & \sum_{i=1}^{m} t_i \\
\sum_{i=1}^{m} t_i & \sum_{i=1}^{m} t_i^2
\end{pmatrix}, \quad x = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \quad y = \begin{pmatrix} \sum_{i=1}^{m} b_i \\
\sum_{i=1}^{m} b_i t_i
\end{pmatrix}
\]
We can write each of these sums as inner products:
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\[
\begin{align*}
\sum_{i=1}^{m} 1 &= 1^T 1, & \sum_{i=1}^{m} t_i &= 1^T t = t^T 1, & \sum_{i=1}^{m} t_i^2 &= t^T t \\
\sum_{i=1}^{m} b_i &= 1^T b = b^T 1, & \sum_{i=1}^{m} b_i t_i &= b^T t = t^T b,
\end{align*}
\]

where

\[
1^T = (1 \cdots 1), \quad t^T = (t_1 \cdots t_m), \quad b^T = (b_1 \cdots b_m)
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\end{align*}
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With this notation we have

\[
Qx = y \iff
\]
We can write each of these sums as inner products:

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\]

\[
\sum_{i=1}^{m} b_i = 1^T b = b^T 1, \quad \sum_{i=1}^{m} b_i t_i = b^T t = t^T b,
\]

where

\[
1^T = (1 \cdots 1), \quad t^T = (t_1 \cdots t_m), \quad b^T = (b_1 \cdots b_m)
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With this notation we have

\[
Qx = y \iff \begin{pmatrix} 1^T 1 & 1^T t \\ t^T 1 & t^T t \end{pmatrix} x = \begin{pmatrix} 1^T b \\ t^T b \end{pmatrix}
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\sum_{i=1}^{m} b_i &= 1^T b = b^T 1, & \sum_{i=1}^{m} b_i t_i &= b^T t = t^T b,
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With this notation we have

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Qx = y \iff \begin{pmatrix}
1^T 1 & 1^T t \\
t^T 1 & t^T t
\end{pmatrix} x = \begin{pmatrix}
1^T b \\
t^T b
\end{pmatrix}
\]

\[
\iff A^T A x = A^T b, \quad A^T = \begin{pmatrix}
1^T \\
t^T
\end{pmatrix}, \quad A = \begin{pmatrix}
1 \\
t
\end{pmatrix}
\]
Therefore we can find the parameters of the line, $x = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$, by solving the square linear system

$$A^T A x = A^T b.$$
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$$A^T A x = A^T b.$$ 

Also note that since $\varepsilon_i = \alpha + \beta t_i - b_i$ we have

$$\varepsilon = \begin{pmatrix} \varepsilon_1 \\ \vdots \\ \varepsilon_m \end{pmatrix} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \alpha + \begin{pmatrix} t_1 \\ \vdots \\ t_m \end{pmatrix} \beta - \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$$

$$= 1\alpha + t\beta - b$$
Therefore we can find the parameters of the line, \( x = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \), by solving the square linear system

\[
A^T A x = A^T b.
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\]
Therefore we can find the parameters of the line, \( \mathbf{x} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \), by solving the square linear system

\[
A^T A \mathbf{x} = A^T \mathbf{b}.
\]

Also note that since \( \varepsilon_i = \alpha + \beta t_i - b_i \) we have

\[
\begin{pmatrix}
\varepsilon_1 \\
\vdots \\
\varepsilon_m
\end{pmatrix} = 
\begin{pmatrix}
1 \\
\vdots \\
1
\end{pmatrix} \alpha + 
\begin{pmatrix}
t_1 \\
\vdots \\
t_m
\end{pmatrix} \beta - 
\begin{pmatrix}
b_1 \\
\vdots \\
b_m
\end{pmatrix} = 1 \alpha + t \beta - b = A \mathbf{x} - \mathbf{b}.
\]

This implies that

\[
G(\alpha, \beta) = \sum_{i=1}^{m} \varepsilon_i^2
\]
Therefore we can find the parameters of the line, \( \mathbf{x} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \), by solving the square linear system

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A^T A \mathbf{x} = A^T \mathbf{b}.
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\]

\[
= 1 \alpha + t \beta - \mathbf{b} = A \mathbf{x} - \mathbf{b}.
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G(\alpha, \beta) = \sum_{i=1}^{m} \varepsilon_i^2 = \varepsilon^T \varepsilon
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Therefore we can find the parameters of the line, $\mathbf{x} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$, by solving the square linear system

$$A^T Ax = A^T b.$$ 

Also note that since $\varepsilon_i = \alpha + \beta t_i - b_i$ we have

$$\mathbf{\varepsilon} = \begin{pmatrix} \varepsilon_1 \\ \vdots \\ \varepsilon_m \end{pmatrix} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \alpha + \begin{pmatrix} t_1 \\ \vdots \\ t_m \end{pmatrix} \beta - \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} = \mathbf{1} \alpha + \mathbf{t} \beta - \mathbf{b} = A \mathbf{x} - \mathbf{b}.$$ 

This implies that

$$G(\alpha, \beta) = \sum_{i=1}^{m} \varepsilon_i^2 = \mathbf{\varepsilon}^T \mathbf{\varepsilon} = (A \mathbf{x} - \mathbf{b})^T (A \mathbf{x} - \mathbf{b}).$$
Example

Data:

<table>
<thead>
<tr>
<th>t</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>b</td>
<td>10</td>
<td>9</td>
<td>7</td>
<td>5</td>
<td>4</td>
<td>3</td>
<td>0</td>
<td>-1</td>
</tr>
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</table>

A \begin{bmatrix} T \\ A \end{bmatrix} x = A \begin{bmatrix} T \\ b \end{bmatrix} \iff \begin{bmatrix} \sum_{i=1}^{8} t_i \sum_{i=1}^{8} t_i^2 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{8} b_i \\ \sum_{i=1}^{8} t_i b_i \end{bmatrix} = \begin{bmatrix} 37 \\ 25 \end{bmatrix} = \Rightarrow \alpha \approx 8.643, \beta \approx -1.607.

So that the best fit line to the given data is

f(t) \approx 8.643 - 1.607 t.
Example

Data:

<table>
<thead>
<tr>
<th>t</th>
<th>-1 0 1 2 3 4 5 6</th>
</tr>
</thead>
<tbody>
<tr>
<td>b</td>
<td>10 9 7 5 4 3 0 -1</td>
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\[ A^T A x = A^T b \]
Example

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\[ A^T A x = A^T b \iff \begin{pmatrix} \sum_{i=1}^{8} 1 & \sum_{i=1}^{8} t_i \\ \sum_{i=1}^{8} t_i & \sum_{i=1}^{8} t_i^2 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^{8} b_i \\ \sum_{i=1}^{8} b_i t_i \end{pmatrix} \]

So that the best fit line to the given data is

\[ f(t) \approx \alpha + \beta t \]

\[ \alpha \approx 8.643, \ \beta \approx -1.607 \]
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\[ A^T A x = A^T b \iff \begin{pmatrix} \sum_{i=1}^{8} 1 & \sum_{i=1}^{8} t_i \\ \sum_{i=1}^{8} t_i & \sum_{i=1}^{8} t_i^2 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^{8} b_i \\ \sum_{i=1}^{8} b_i t_i \end{pmatrix} \]

\[ \iff \begin{pmatrix} 8 & 20 \\ 20 & 92 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 37 \\ 25 \end{pmatrix} \]

So that the best fit line to the given data is

\[ f(t) \approx 8.643 - 1.607 t \]
Example

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<tr>
<th>t</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>b</td>
<td>10</td>
<td>9</td>
<td>7</td>
<td>5</td>
<td>4</td>
<td>3</td>
<td>0</td>
<td>-1</td>
</tr>
</tbody>
</table>

\[
A^T A x = A^T b \iff \begin{pmatrix}
\sum_{i=1}^{8} 1 & \sum_{i=1}^{8} t_i \\
\sum_{i=1}^{8} t_i & \sum_{i=1}^{8} t_i^2
\end{pmatrix}
\begin{pmatrix}
\alpha \\
\beta
\end{pmatrix}
= \begin{pmatrix}
\sum_{i=1}^{8} b_i \\
\sum_{i=1}^{8} b_i t_i
\end{pmatrix}
\iff
\begin{pmatrix}
8 & 20 \\
20 & 92
\end{pmatrix}
\begin{pmatrix}
\alpha \\
\beta
\end{pmatrix}
= \begin{pmatrix}
37 \\
25
\end{pmatrix}
\implies \alpha \approx 8.643, \beta \approx -1.607

So that the best fit line to the given data is

\[
f(t) \approx 8.643 - 1.607 t
\]
Example

Data:

<table>
<thead>
<tr>
<th>t</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
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<td>-1</td>
</tr>
</tbody>
</table>

\[ A^T A x = A^T b \]

\[ \begin{pmatrix} \sum_{i=1}^{8} 1 \\ \sum_{i=1}^{8} t_i \\ \sum_{i=1}^{8} t_i^2 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^{8} b_i \\ \sum_{i=1}^{8} b_i t_i \end{pmatrix} \]

\[ \begin{pmatrix} 8 & 20 \\ 20 & 92 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 37 \\ 25 \end{pmatrix} \]

\[ \Rightarrow \alpha \approx 8.643, \quad \beta \approx -1.607 \]

So that the best fit line to the given data is

\[ f(t) \approx 8.643 - 1.607t. \]
General Least Squares

The general least squares problem behaves analogously to the linear example.

**Theorem**

Let $A$ be a real $m \times n$ matrix and $b$ an $m$-vector. Any vector $x$ that minimizes the square of the residual $Ax - b$, i.e.,

$$G(x) = (Ax - b)^T(Ax - b)$$

is called a least squares solution of $Ax = b$. 
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\[
G(x) = (Ax - b)^T (Ax - b)
\]

is called a least squares solution of \( Ax = b \).

The set of all least squares solutions is obtained by solving the normal equations

\[
A^T Ax = A^T b.
\]

Moreover, a unique solution exists if and only if \( \text{rank}(A) = n \) so that

\[
x = (A^T A)^{-1} A^T b.
\]
Proof

The statement about \textit{uniqueness follows directly from our earlier theorem} on p. 92 on the normal equations.
Proof

The statement about uniqueness follows directly from our earlier theorem on p. 92 on the normal equations.

To characterize the least squares solutions we first show that if \( x \) minimizes \( G(x) \) then \( x \) satisfies \( A^T Ax = A^T b \).
Proof

The statement about uniqueness follows directly from our earlier theorem on p. 92 on the normal equations.

To characterize the least squares solutions we first show that if $x$ minimizes $G(x)$ then $x$ satisfies $A^T Ax = A^T b$.

As in our earlier example, a necessary condition for the minimum is:

$$\frac{\partial G(x)}{\partial x_i} = 0, \ i = 1, \ldots, n.$$
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Let’s first work out what $G(x)$ looks like:

$$G(x) = (Ax - b)^T (Ax - b)$$
Classical Least Squares

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\[
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\]

Let’s first work out what \( G(x) \) looks like:

\[
G(x) = (Ax - b)^T (Ax - b) = x^T A^T A x - 2 x^T A^T b + b^T b - x^T A^T b - b^T A x + b^T b
\]
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$$= x^T A^T Ax - x^T A^T b - b^T A x + b^T b$$

$$= x^T A^T Ax - 2x^T A^T b + b^T b$$

since $b^T A x = (b^T A x)^T = x^T A^T b$ is a scalar.
Therefore

\[
\frac{\partial G(x)}{\partial x_i} = \frac{\partial x^T}{\partial x_i} A^T Ax + x^T A^T A \frac{\partial x}{\partial x_i} - 2 \frac{\partial x^T}{\partial x_i} A^T b
\]
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\frac{\partial G(x)}{\partial x_i} = \frac{\partial x^T}{\partial x_i} A^T A x + x^T A^T A \frac{\partial x}{\partial x_i} - 2 \frac{\partial x^T}{\partial x_i} A^T b
\]

\[
= e_i^T A^T A x + x^T A^T A e_i - 2 e_i^T A^T b
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\[
= e_i^T A^T A x + x^T A^T A e_i - 2 e_i^T A^T b
\]

\[
= 2 e_i^T A^T A x - 2 e_i^T A^T b
\]
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\[
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(cont.)

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This means that

\[
\frac{\partial G(x)}{\partial x_i} = 0 \iff \]

fasshauer@iit.edu
Therefore

\[ \frac{\partial G(x)}{\partial x_i} = \frac{\partial x^T}{\partial x_i} A^T A x + x^T A^T A \frac{\partial x}{\partial x_i} - 2 \frac{\partial x^T}{\partial x_i} A^T b \]

\[ = e_i^T A^T A x + x^T A^T A e_i - 2 e_i^T A^T b \]

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This means that

\[ \frac{\partial G(x)}{\partial x_i} = 0 \quad \iff \quad (A^T)_{i*} A x = (A^T)_{i*} b. \]
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= e_i^T A^T A x + x^T A^T A e_i - 2 e_i^T A^T b \\
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If we collect all such conditions (for \( i = 1, \ldots, n \)) in one linear system we get

\[
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To verify that we indeed have a minimum we show that if $z$ is a solution of the normal equations then $G(z)$ is minimal.
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\[ G(z) = (Az - b)^T(Az - b) = z^T A^T A z - 2 z^T A^T b + b^T b \]
(cont.)

To verify that we indeed have a minimum we show that if \( z \) is a solution of the normal equations then \( G(z) \) is minimal.

\[
G(z) = (Az - b)^T (Az - b)
= z^T A^T Az - 2z^T A^T b + b^T b
= z^T (A^T Az - A^T b) - z^T A^T b + b^T b
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\]

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\[
G(y) = (z + u)^T A^T A(z + u) - 2(z + u)^T A^T b + b^T b \\
= G(z) + u^T A^T A u + z^T A^T A u + u^T A^T A z - 2u^T A^T b
\]
(cont.)

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\]

\[
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\[
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\]

\[
= G(z) + u^T A^T Au + z^T A^T Au + u^T A^T Az - 2u^T A^T b
\]

\[
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\]
(cont.)

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= G(z) + u^T A^T Au + z^T A^T Au + u^T A^T Az - 2 u^T A^T b
\]
\[
= u^T A^T Az
\]
\[
ge G(z) + u^T A^T Au \ge G(z)
\]

since $u^T A^T Au = \sum_{i=1}^{m} (Au)_i^2 \ge 0$. □
Remark

*Using this framework we can compute least squares fits from any linear function space.*
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Example

1. Let $f(t) = \alpha_0 + \alpha_1 t + \alpha_2 t^2$, i.e., we can use quadratic polynomials (or any other degree).
Remark

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1. Let \( f(t) = \alpha_0 + \alpha_1 t + \alpha_2 t^2 \), i.e., we can use quadratic polynomials (or any other degree).
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2. Let $f(t) = \alpha_0 + \alpha_1 \sin t + \alpha_2 \cos t$, i.e., we can use trigonometric polynomials.
3. Let $f(t) = \alpha e^t + \beta \sqrt{t}$, i.e., we can use just about anything we want.
Regression in Statistics (BLUE)

One assumes that there is a random process that generates data as a random variable $Y$ of the form

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \ldots + \beta_n X_n,$$

where $X_1, \ldots, X_n$ are (input) random variables and $\beta_1, \ldots, \beta_n$ are unknown parameters.
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where $X_1, \ldots, X_n$ are (input) random variables and $\beta_1, \ldots, \beta_n$ are unknown parameters.

Now the actually observed data may be affected by noise, i.e.,

$$y = Y + \varepsilon = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \ldots + \beta_n X_n + \varepsilon,$$

where $\varepsilon \sim \mathcal{N}(0, \sigma^2)$ (normally distributed with mean zero and variance $\sigma^2$) is another random variable denoting the noise.
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where $\varepsilon \sim \mathcal{N}(0, \sigma^2)$ (normally distributed with mean zero and variance $\sigma^2$) is another random variable denoting the noise.

To determine the model parameters $\beta_1, \ldots, \beta_n$ we now look at measurements, i.e.,

$$y_i = \beta_0 + \beta_1 x_{i,1} + \beta_2 x_{i,2} + \ldots + \beta_n x_{i,n} + \varepsilon, \quad i = 1, \ldots, m.$$
In matrix-vector form this gives us

\[ y = X\beta + \varepsilon \]

Now, the least squares solution of \( X\beta = y \), i.e., \( \hat{\beta} = (X^TX)^{-1}X^Ty \) is in fact the best linear unbiased estimator (BLUE) for \( \beta \).
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Then

\[ \mathbb{E}[y] = \mathbb{E}[X\beta + \epsilon] \]
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Then

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and therefore

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In matrix-vector form this gives us

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Now, the least squares solution of \( \mathbf{X} \beta = \mathbf{y} \), i.e., \( \hat{\beta} = (\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \mathbf{y} \) is in fact the best linear unbiased estimator (BLUE) for \( \beta \).

To show this one needs an assumption that the error is unbiased, i.e., \( \mathbb{E}[\mathbf{\varepsilon}] = \mathbf{0} \).

Then

\[ \mathbb{E}[\mathbf{y}] = \mathbb{E}[\mathbf{X} \beta + \mathbf{\varepsilon}] = \mathbb{E}[\mathbf{X} \beta] + \mathbb{E}[\mathbf{\varepsilon}] = \mathbf{X} \beta \]

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\[ = (\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \mathbf{X} \beta \]
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Then

\[ \mathbb{E}[y] = \mathbb{E}[X\beta + \epsilon] = \mathbb{E}[X\beta] + \mathbb{E}[\epsilon] = X\beta \]

and therefore

\[ \mathbb{E}[\hat{\beta}] = \mathbb{E}[(X^T X)^{-1} X^T y] = (X^T X)^{-1} X^T \mathbb{E}[y] \]

\[ = (X^T X)^{-1} X^T X\beta = \beta, \]

so that the estimator is indeed unbiased.
Remark

One can also show (maybe later) that \( \hat{\beta} \) has minimal variance among all unbiased linear estimators, so it is the best linear unbiased estimator of the model parameters.

In fact, the theorem ensuring this is the so-called Gauss-Markov theorem.
Outline

1. Spaces and Subspaces
2. Four Fundamental Subspaces
3. Linear Independence
4. Bases and Dimension
5. More About Rank
6. Classical Least Squares
7. Kriging as best linear unbiased predictor
Kriging: a regression approach

**Assume:** the approximate value of a realization of a zero-mean (Gaussian) random field is given by a linear predictor of the form

\[
\hat{Y}_x = \sum_{j=1}^{N} Y_{x_j} w_j(x) = w(x)^T Y,
\]

where \( \hat{Y}_x \) and \( Y_{x_j} \) are random variables, \( Y = (Y_{x_1} \cdots Y_{x_N})^T \), and \( w(x) = (w_1(x) \cdots w_N(x))^T \) is a vector of weight functions at \( x \).
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We now present some details (see [FM15]).
Covariance Kernel

We need the covariance kernel $K$ of a random field $Y$ with mean $\mu(x)$. It is defined via

$$\sigma^2 K(x, z) = \text{Cov}(Y_x, Y_z) = \mathbb{E}[(Y_x - \mu(x))(Y_z - \mu(z))]$$
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Therefore, the variance of the random field,

$$\text{Var}(Y_x) = \mathbb{E}[Y_x^2] - \mathbb{E}[Y_x]^2 = \mathbb{E}[Y_x^2] - \mu^2(x),$$

corresponds to the “diagonal” of the covariance, i.e.,

$$\text{Var}(Y_x) = \sigma^2 K(x, x).$$
Let’s now work out the MSE:

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Now use \( \mathbb{E}[Y_x Y_z] = K(x, z) \) (the covariance, since \( Y \) is centered):

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\text{MSE}(\hat{Y}_x) = \sigma^2 K(x, x) - 2w(x)^T (\sigma^2 k(x)) + w(x)^T (\sigma^2 K) w(x),
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where

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\sigma^2 k(x) = \sigma^2 \begin{pmatrix} k_1(x) & \cdots & k_N(x) \end{pmatrix}^T: \text{ with }
\]

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\( K: \) the covariance matrix has entries \( \sigma^2 K(x_i, x_j) = \mathbb{E}[Y_{x_i} Y_{x_j}] \)
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and so the optimum weight vector is

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\hat{w}(x) = K^{-1} k(x).
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Since we are given the observations $y$ as realizations of $Y$ we can compute the prediction

$$\hat{y}_x = k(x)^T K^{-1} y.$$
The MSE of the kriging predictor with optimal weights $\hat{w}^*(\cdot)$,

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\mathbb{E} \left[ (Y_x - \hat{Y}_x)^2 \right] = \sigma^2 \left( K(x, x) - k(x)^T K^{-1} k(x) \right),
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is known as the kriging variance.

It allows us to give confidence intervals for our prediction. It also gives rise to a criterion for choosing an optimal parametrization of the family of covariance kernels used for prediction.
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**Remark**

For Gaussian random fields the BLUP is also the best nonlinear unbiased predictor (see, e.g., [BTA04, Chapter 2]).
Remark

1. The simple kriging approach just described is precisely how Krige [Kri51] introduced the method:
   - The unknown value to be predicted is given by a weighted average of the observed values, where the weights depend on the prediction location.
   - Usually one assigns a smaller weight to observations further away from $x$. 
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The latter statement implies that one should be using kernels whose associated weights decay away from $x$. Positive definite translation invariant kernels have this property.
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More advanced kriging variants are discussed in papers such as [SWMW89, SSS13], or books such as [Cre93, Ste99, BTA04].
References I


References II
