SHOW ALL WORK! USE THESE SHEETS ONLY.

1. Write down the linear Lagrange interpolation polynomial for the function $f(x) = x^3$ using the points $x_0 = 0$ and $x_1 = b$. Verify formula (7) in Theorem 1.9 of the class notes by direct calculation. In particular, show that in this case $\xi$ has the unique value $\xi = \frac{1}{3}(x + b)$.

Data:
\[
\begin{array}{c|c|c}
\quad & x & y \\
\hline
0 & 0 & 0 \\
\hline
b & b^3 & \\
\end{array}
\]

So Lagrange polynomial is
\[
p_i(x) = \sum_{j=0}^{1} l_j(x) f(x_j) = l_1(x) b^3 = \frac{x-x_0}{x_1-x_0} b^3 = \frac{x}{b} b^3 = b x
\]

\[\text{Difference:} \quad f(x) - p(x) = x^3 - b^2 x = x(x^2 - b^2) = x(x-b)(x+b) \quad (1)\]

Estimate from Thm 1.9:
\[
f(x) - p(x) = \frac{1}{2!} f''(\xi) (x-x_0)(x-x_1)
\]
\[
= \frac{1}{2} 6\xi x (x-b) \quad \text{(since } f''(x) = 6x) \quad \text{(since } f''(x) = 6x) \quad (2)
\]

Comparing (1) and (2) we see
\[
3\xi = x+b \quad \text{or} \quad \xi = \frac{1}{3}(x+b)
\]
2. (a) Find a formula of the form \[
\int_0^{2\pi} f(x) \, dx = A_1 f(0) + A_2 f(\pi) \]
that is exact for any function having the form \( f(x) = a \cos x + b \).

(b) Prove that the formula derived in (a) is exact for any function of the form

\[
f(x) = \sum_{k=0}^{n} [a_k \cos(2k+1)x + b_k \sin kx].
\]

(b) Since \( 1, \cos x \) are lin independent on \([0, 2\pi]\), we can use them as basis and check

\[
\int_0^{2\pi} dx = 2\pi = A_1 + A_2
\]

and

\[
\int_0^{2\pi} \cos x \, dx = 0 = A_1 - A_2 \]

\[
\Rightarrow \quad A_1 = A_2 = \frac{\pi}{2}
\]

(b) Again, \( \{\cos (2k+1)x, \sin kx, k = 0, \ldots, n\} \) are linearly independent on \([0, 2\pi]\).

So check only if exact for them:

\[
\int_0^{2\pi} \cos (2k+1)x \, dx = 0 = A_1 - A_2 = \pi - \pi = 0 \quad \checkmark
\]

\[
\int_0^{2\pi} \sin kx \, dx = 0 = OA_1 + OA_2
\]
3. Is it possible to find coefficients \( a, b, c, \) and \( d \) such that the function

\[
S(x) = \begin{cases} 
1 - 2x, & x \leq -3, \\
ax + bx + cx^2 + dx^3, & -3 < x \leq 4, \\
157 - 32x, & 4 < x < \infty, 
\end{cases}
\]

is a natural cubic spline for the interval \([-3, 4]\)?

\( C^0 \) continuity:
\[
S_0(-3) = S_1(-3) = a - 3b + 9c - 27d = 7 \quad (1)
\]
\[
S_1(4) = a + 4b + 16c + 64d = S_2(4) = 29 \quad (2)
\]

\( C^1 \) continuity:
\[
S_0'(-3) = S_1'(-3) = b + 2c + 3dx^2 \bigg|_{x=-3} = -2 = b - 6c + 27d \quad (3)
\]
\[
S_1'(4) = b + 2c + 3dx^2 \bigg|_{x=4} = b + 8c + 48d = S_2'(4) = -32 \quad (4)
\]

\( C^2 \) continuity:
\[
S_0''(-3) = S_1''(-3) = 2c + 6dx \bigg|_{x=-3} = 2c - 18d \quad (5)
\]
\[
S_1''(4) = 2c + 6dx \bigg|_{x=4} = 2c + 24d \bigg|_{x=4} = 0 = S_2''(4) \quad (6)
\]

From (5) and (6) we have \( c = d = 0 \)

but then (3) \( \Rightarrow \) \(-2 = b \) and (4) \( \Rightarrow \) \(-32 = b \) \( \rightarrow \) a contradiction not possible
(a) Apply Richardson extrapolation to Euler's method using step sizes $h$ and $h/2$ to derive the second-order Runge-Kutta method (modified Euler method) $y_{n+1} = y_n + h k_2$, where

$$k_1 = f(t_n, y_n), \quad k_2 = f(t_n + \frac{h}{2}, y_n + \frac{h}{2} k_1).$$

(b) How can this method be used to produce a table of values for the function $f(x) = \int_0^x e^{-t^2} dt$ at 100 equally spaced points on $[0,1]$?

(a) Euler: $y_{n+1} = y_n + hf(t_n, y_n)$ is $O(h)$.

Richardson: if $F = F_h + O(h^p)$, then

$$F \approx \frac{2^p}{2p+1} \left[ F_{h/2} - F_h \right], \quad p = 1$$

$F_h$: $y_{n+1} = y_n + hf(t_n, y_n)$

$F_{h/2}$: $\tilde{y}_{n+1} = y_n + \frac{h}{2} f(t_n, y_n)$

and $y_{n+1} = \tilde{y}_{n+1} + \frac{h}{2} f(t_{n+1}, \tilde{y}_{n+1}) = y_n + \frac{h}{2} f(t_n, y_n) + \frac{h}{2} f(t_n + \frac{h}{2}, y_n + \frac{h}{2} k_1)$

$$= y_n + \frac{h}{2} \left[ \frac{f(t_n, y_n) + f(t_n + \frac{h}{2}, y_n + \frac{h}{2} k_1)}{k_1} \right]$$

$$= y_n + \frac{h}{2} \left[ k_1 + k_2 \right]$$

Now combine

$$2 F_{h/2} - F_h = 2 \left[ y_n + \frac{h}{2} (k_1 + k_2) \right] - (y_n + k_1) = y_n + h k_2$$

modified Euler.
(b) Computation of $\int_0^x e^{-t^2} \, dt$ is equivalent to solving (fund. theorem of calculus)

$$y'(t) = e^{-t^2}$$

$$y(0) = 0$$

So just apply modified Euler with $f(t, y) = e^{-t^2}$ and $h = \frac{1}{99}$
5. By considering the scalar model initial value problem

\[ y'(t) = \lambda y(t), \quad t \in [0, T], \]
\[ y(0) = y_0, \]

with real \( \lambda \) determine the linear stability domain (in this case a real interval) for the modified Euler method of Problem 4.

Use modified Euler with \( f(t, y) = \lambda y(t) \)

so

\[ y_{n+1} = y_n + h \left( y_n + \frac{h}{2} \frac{y_n}{k_1} \right) = y_n + h \lambda \left( y_n + \frac{h}{2} \frac{y_n}{k_1} \right) \]

\[ = y_n + h \lambda y_n + \frac{h^2 \lambda^2}{2} y_n = \left( 1 + h \lambda + \frac{(h \lambda)^2}{2} \right) y_n \]

Apply recursively so that

\[ y_n = \left( 1 + h \lambda + \frac{(h \lambda)^2}{2} \right)^n y_0 \]

For stability we need growth factor

\[ \left| 1 + h \lambda + \frac{(h \lambda)^2}{2} \right| < 1 \]

\[ -1 < 1 + h \lambda + \frac{(h \lambda)^2}{2} < 1 \]

\[ \Rightarrow h \lambda + \frac{(h \lambda)^2}{2} < 0 \quad \text{and} \quad h \lambda + \frac{(h \lambda)^2}{2} > 0 \]

\[ \Rightarrow (z+1)(z+4) > 0 \quad \Rightarrow \quad -2 < z < 0 \]

or \( 0 < h \lambda < -\frac{2}{\lambda} \) (since \( \lambda < 0 \))