Friday, September 19, 2014

Let f be a function with the following properties:

1. $f(n)$ is defined for every positive integer n.
2. $f(n)$ is an integer.
3. $f(2) = 2$.
4. $f(m\,n) = f(m)f(n)$ for all m, n.
5. $f(m) > f(n)$ for $n = 1, 2, 3, \ldots$

Prove that $f(n) = n$ for $n = 1, 2, 3, \ldots$

Solution. We argue by induction. Since $f(1)$ exists and is an integer such that $f(1) = (f(1))^2$ we must have that $f(1) = 0$ or $f(1) = 1$, but $f(1)$ can't be equal to zero as this and property 4 would imply that property 5 is not true. So $f(1) = 1$.

Now suppose that for $k \leq n$ we have that $f(k) = k$. We claim that $f(n+1) = f(n+1)$. There are two cases:

- Case 1: $\underline{n+1 = 2j}$ then $f(n+1) = f(2j) = f(2)f(j) = 2j = n+1$.
- Case 2: $\underline{n+1 = 2j+1}$ then $1 \leq j < n$ and

  $$2j = f(2j) < f(2j+1) < f(2j+2) = f(2(j+1)) = f(2)f(j+1) = 2(j+1) = 2j+2$$

  So $f(n+1) = f(2j+1) = 2j+1 = n+1$.

Good Luck! Have fun and enjoy Mathematics!