Friday, October 24, 2014

Find all positive integers n such that $2^4 + 2^7 + 2^n$ is a perfect square.

Solution. The only solution is $n = 8$.

Suppose that $2^4 + 2^7 + 2^n = a^2$. First, consider the case that $n \geq 4$. Then $2^4 = 16$ divides the left side, so 16 must also divide a^2. Then 4 divides a; let $b = a/4$, an integer. Factoring out 16 of both sides, letting $m = n - 4$, we get $16(1 + 8 + 2^m) = 16b^2$, or $2^m = b^2 - 9$. Then $2^m = (b + 3)(b - 3)$. Then each of $b + 3$ and $b - 3$ must be a power of 2. One can quickly check that the only powers of 2 that differ by 6 are 2 and 8, so $b + 3 = 8$ and $b - 3 = 2$. Then $2^m = 8 \cdot 2 = 16$, so $m = 4$ and $n = 4 + 4 = 8$.

What about if $n < 4$? $2^4 + 2^7 + 2^n$ is even since n is positive. Then a^2 is even, so a is also even. Further a^2 is not divisible by 16, so a is not divisible by 4. Therefore $a = 2(2c + 1)$ where c is an integer. Then $2^4 + 2^7 + 2^n = 4(4c^2 + 4c + 1)$. Dividing through by 4, we get $2^2 + 2^5 + 2^{n-2} = 4c^2 + 4c + 1$. Then it must be that $2^{n-2} = 1$, since the other terms are even. Subtracting 1 and then factoring 4 out, we get $2^0 + 2^3 = c^2 + c$, or $9 = c(c + 1)$. The right side must be even, so there is no solution.

Good Luck! Have fun and enjoy Mathematics!