Friday, November 21, 2014

Suppose that $(f(x) + f'(x)) \to L$ as $x \to \infty$. Show that $f(x) \to L$ as $x \to \infty$ (and hence $f'(x) \to 0$). You may also assume that $f'(x)$ is continuous.

Solution. Replacing $f(x)$ by $(f(x) - L)$, we can assume that $L = 0$. If $\lim_{x\to\infty} f(x) = 0$, we are done. So suppose that $\lim_{x\to\infty} f(x) \neq 0$ and let

$$C = \{x : x > 0, f'(x) = 0\} .$$

We shall show that C is bounded above. Suppose to the contrary that C is not bounded above. Then $\lim_{x\to\infty, x\in C} f(x) = 0$. Since $\lim_{x\to\infty} f(x) \neq 0$, there exists $\epsilon > 0$ and a sequence $\{y_n\}$ with $y_n \to \infty$ and $|f(y_n)| > \epsilon$. Choose N such that if $x \in C$ and $x > N$ then $|f(x)| < \epsilon$. Choose $x_1 \in C$, $x_1 > N$. Choose $y_n > x_1$. Choose $x_2 \in C$, $x_2 > y_n$. Consider

$$\max\{f(x) : x_1 \le x \le x_2\} \qquad \text{and} \qquad \min\{f(x) : x_1 \le x \le x_2\}$$

These exist by the Extreme Value Theorem and must occur at some point in C (since the endpoints of the interval are in C). However $|f(x)| < \epsilon$ at all these points and $|f(y_n)| > \epsilon$, a contradiction. Thus C is bounded above.

Since C is bounded above, there exists N such that $f'(x) > 0$ for all $x > N$ or $f'(x) < 0$ for all $x > N$. This is true by the Intermediate Value Theorem under the assumption that $f'(x)$ is continuous (it is actually true regardless of whether we assume $f'(x)$ is continuous by Darboux's Theorem). Multiplying by (-1) if necessary, we assume $f'(x) > 0$ for all $x > N$. So $f(x)$ is strictly increasing for $x > N$. Thus $\lim_{x\to\infty} f(x) = K$ exists or is $+\infty$.

Since $(f(x) + f'(x)) \to 0$ as $x \to \infty$ and $f'(x) > 0$, we have $K \le 0$. If $K < 0$, then $f'(x) \to -K$ as $x \to \infty$ and hence there is M such that $f'(x) > -K/2 > 0$ if $x > M$ and then we would have $f(x) \to \infty$ as $x \to \infty$. Therefore $K = 0$.

Good Luck! Have fun and enjoy Mathematics!