Friday, November 8, 2013

Prove that if you add up the reciprocals of a sequence of consecutive positive integers, the numerator of the sum (in lowest terms) will always be odd. For example, $\frac{1}{7} + \frac{1}{8} + \frac{1}{9} = \frac{191}{504}$.

Solution. Any integer can be written as a power of 2 multiplied by an odd integer (divide out 2's until you cannot): $2^a b$. Let 2^k be the largest power of two that divides one of the integers n in the sequence. Then $n = 2^k j$, where j is odd. The next largest and next smallest integers which are multiples of 2^k are $2^k(j+1)$ and $2^k(j-1)$, both of which are divisible by 2^{k+1} (since $j+1$ and $j-1$ are even), so neither of these are in the sequence. Therefore, n is the only integer in the list which is a multiple of 2^k.

To sum up the fractions, we need a common denominator. For each of the reciprocals $\frac{1}{2^a b}$, multiply the numerator and denominator by 2^{k-a}. Every term's denominator is now an odd multiple of 2^k, and the numerators are all even except for $1/n$ (which is unchanged). Now, to find a common denominator, the numerators and denominators of each term must be multiplied by odd numbers; afterwards, it is still the case that all but one of the numerators are even. Now that every denomnator is the same, we add together all the numerators, and since exactly one is odd, their sum is odd.

If it is not yet in lowest terms yet, we divide the numerator and denominator by the same (odd) number, and the resulting numerator will still be odd.

Good Luck! Have fun and enjoy Mathematics!