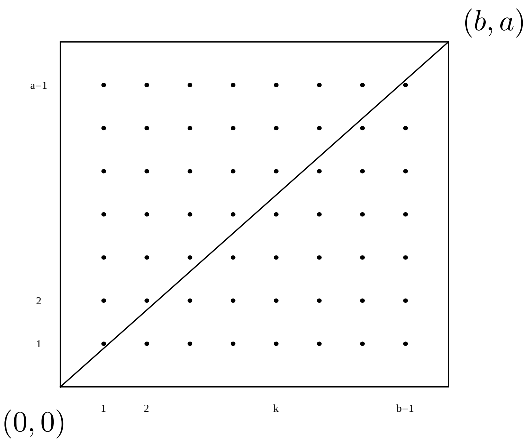
Friday, January 31, 2014

Let $\lfloor x \rfloor$ be the *floor of* x, i.e., the greatest integer less than or equal to x. Now, assume a and b are positive integers with no common factor. Show that

$$\left\lfloor \frac{a}{b} \right\rfloor + \left\lfloor \frac{2a}{b} \right\rfloor + \cdots + \left\lfloor \frac{(b-1)a}{b} \right\rfloor = \frac{(a-1)(b-1)}{2}.$$

Solution. Consider the rectangle in the Cartesian plane with opposite corners at $(0,0)$ and (b,a) and the diagonal connecting these corners (which lies on the line $y = ax/b$). Further consider the lattice points (m,n) whose coordinates are integers within the interior of this rectangle.



For each $k = 1, 2, \ldots, b-1$, $\left\lfloor \frac{ka}{b} \right\rfloor$ is precisely equal to the number of such lattice points, whose x-coordinate is k, that are on or below the line $y = \frac{ax}{b}$. However, the fact that a and b share no common factors implies that line $y = \frac{bx}{a}$ does not intersect any of the lattice points (if $\frac{ak}{b}$ is an integer then b must divide k, but this is impossible if $0 < k < b$). Thus the sum given in the problem is equal to half of the lattice points in the rectangle, i.e., $\dfrac{(a-1)(b-1)}{2}$.

Good Luck! Have fun and enjoy Mathematics!