Friday, January 24, 2014

Find positive integers n and $a_1, a_2, \ldots a_n$ such that their product $a_1 a_2 \cdots a_n$ is maximized and their sum $a_1 + a_2 + \cdots + a_n = 2014$.

Solution. Suppose that n and $a_1, a_2, \ldots a_n$ are chosen so that $a_1 + a_2 + \cdots + a_n = 2014$ and the product $a_1 a_2 \cdots a_n$ is maximum.

Claim 1: None of the terms equal 1. Clearly if this were the case then the product could be increased by removing that term and increasing the value of another term by 1.

Claim 2: None of the terms are odd and greater than 3. If a term was of the form $2k + 1$ for some integer $k \geq 2$, then replacing this term with the two terms k and $k + 1$ would increase the product.

Claim 3: None of the terms are even and greater than 4. If some term was of the form $2k$ with $k > 2$, then replacing this term with the two terms k and k will increase the product.

Claim 4: If any of the terms equal 4, then the product will be unchanged by replacing this term by the two terms each equal to 2.

Claim 5: There cannot be three terms each equal to 2. If there were, then the product could be increased by replacing these three terms with the two terms 3 and 3.

It follows from these claims is that either

- $n = 672$ and the terms consist of six hundred and seventy 3's and two 2's, or
- $n = 671$ and the terms consist of six hundred and seventy 3's and one 4

and that the product $a_1 a_2 \cdots a_n = 3^{670} 2^2$.

Good Luck! Have fun and enjoy Mathematics!