Friday, February 27, 2015

Let

\[ a_n = \frac{1^2 + 2^2 + \cdots + n^2}{n^2}. \]

Show that \( \lim_{n \to \infty} a_n \) exists and find its value.

**Solution.** The limit is \( \frac{2}{9} \). Write \( a_n \) in the form

\[ a_n = \frac{1}{n} \left[ \left( \frac{1}{n} \right)^2 + \left( \frac{2}{n} \right)^2 + \cdots + \left( \frac{n}{n} \right)^2 \right], \]

and recognize this as a Riemann sum for the integral

\[ \int_0^1 x^2 \, dx = \frac{2}{9} x^3 \bigg|_0^1 = \frac{2}{9}. \]

Good Luck! Have fun and enjoy Mathematics!