ILLINOIS INSTITUTE OF TECHNOLOGY Department of Applied Mathematics and IIT SIAM Student Chapter

Math Weekly Problem Competition

Friday, February 14, 2014

Evaluate the infinite product $\prod_{k=1}^{\infty} \cos(x2^{-k})$.

(Note the infinite product is, by definition, the limit of the partial products

$$P_n = \prod_{k=1}^n \cos(x2^{-k}) = \cos(x2^{-1})\cos(x2^{-2})\cdots\cos(x2^{-n})$$

as $n \to \infty$.)

Solution. If x = 0, then each $P_n = 1$ and hence $\lim_{n \to \infty} P_n = 1$. For the case of $x \neq 0$: The trigonometric identity $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$ and induction yields that

$$\sin(2^n\theta) = 2^n \sin(\theta) \cos(\theta) \cos(2\theta) \cdots \cos(2^{n-1}\theta) .$$

Applying this identity with $\theta = x \cdot 2^{-n}$ gives that

$$P_n = \frac{\sin x}{2^n \sin(x 2^{-n})}$$

(provided n is large enough so that $x2^{-n}$ is not a multiple of π). Since, by l'Hopital's rule, $\lim_{y\to\infty}(\sin y)/y = 1$, we have $\lim_{n\to\infty} 2^n \sin(x2^{-n}) = x$. Hence the value of the given product is $\lim_{n\to\infty} P_n = \frac{\sin(x)}{x}$ if $x \neq 0$.

Good Luck! Have fun and enjoy Mathematics!

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