Friday, February 14, 2014

Evaluate the infinite product $\prod_{k=1}^{\infty} \cos(x2^{-k})$.

(Note the infinite product is, by definition, the limit of the partial products

$$P_n = \prod_{k=1}^{n} \cos(x2^{-k}) = \cos(x2^{-1})\cos(x2^{-2})\cdots\cos(x2^{-n})$$

as $n \to \infty$.)

Solution. If $x = 0$, then each $P_n = 1$ and hence $\lim_{n\to\infty} P_n = 1$.
For the case of $x \neq 0$: The trigonometric identity $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$ and induction yields that

$$\sin(2^n\theta) = 2^n \sin(\theta)\cos(\theta)\cos(2\theta)\cdots\cos(2^{n-1}\theta)\ .$$

Applying this identity with $\theta = x \cdot 2^{-n}$ gives that

$$P_n = \frac{\sin x}{2^n \sin(x2^{-n})}$$

(provided n is large enough so that $x2^{-n}$ is not a multiple of π). Since, by l'Hopital's rule, $\lim_{y\to\infty}(\sin y)/y = 1$, we have $\lim_{n\to\infty} 2^n \sin(x2^{-n}) = x$. Hence the value of the given product is $\lim_{n\to\infty} P_n = \sin(x)/x$ if $x \neq 0$.

Good Luck! Have fun and enjoy Mathematics!