Friday, February 08, 2013

Let us consider the polynomial

\[ x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_n, \]

and assume that it has only real roots.

Prove that

\[ (n-1)a_1^2 \geq 2na_2. \]

Find necessary and sufficient conditions such that the last inequality becomes identity.

**Solution.** Since all the roots are real, the polynomial can be represented as

\[ (x + r_1)(x + r_2) \ldots (x + r_n) \]

for some real numbers \( r_1, \ldots, r_n \). Then,

\[ a_1 = \sum_{k=1}^{n} r_k, \quad a_2 = \sum_{1 \leq i < j \leq n} r_i r_j. \]

This implies that

\[ a_1^2 = \sum_{i=1}^{n} r_i^2 + 2a_2. \]

Hence the inequality of interest is equivalent to

\[ (n-1) \left( \sum_{i=1}^{n} r_i^2 + 2a_2 \right) \geq 2na_2, \]

or

\[ (n-1) \sum_{i=1}^{n} r_i^2 \geq 2 \sum_{1 \leq i < j \leq n} r_i r_j, \]

which is equivalent to

\[ \sum_{1 \leq i < j \leq n} (r_i - r_j)^2 \geq 0, \]

which obviously holds true. Moreover, the equality is achieved if and only if all roots are equal.

Good Luck! Have fun and enjoy Mathematics!