Friday, February 6, 2015

Suppose that $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$ are positive real numbers. Show that

$$\frac{a_1}{b_1} + \frac{a_2}{b_2} + \frac{a_n}{b_n} \geq n \qquad \text{or} \qquad \frac{b_1}{a_1} + \frac{b_2}{a_2} + \frac{b_n}{a_n} \geq n\ .$$

Solution. The key fact is that each $\frac{a_i}{b_i} + \frac{b_i}{a_i} \geq 2$. This is because $x + \frac{1}{x} \geq 2$ for all $x > 0$, which is true since this inequality is equivalent to $(x-1)^2 \geq 0$.

From the fact that each $\frac{a_i}{b_i} + \frac{b_i}{a_i} \geq 2$ it follows that

$$\left(\frac{a_1}{b_1} + \frac{a_2}{b_2} + \frac{a_n}{b_n}\right) + \left(\frac{b_1}{a_1} + \frac{b_2}{a_2} + \frac{b_n}{a_n}\right) \geq 2n,$$

and therefore at least one of the sums in the parentheses is $\geq n$.

Good Luck! Have fun and enjoy Mathematics!