Coupling of finite elements discretises ‘SPDE’s

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Abstract

Stochastic centre manifold theory provides novel support for coarse grained, macroscale, spatial discretisations of nonlinear stochastic partial differential equations such as the example of the stochastically forced Burgers’ equation. Dividing the physical domain into finite length overlapping elements empowers the approach to resolve fully coupled dynamical interactions between neighbouring elements. I explore, compare and contrast different methods for coupling the finite elements. The techniques developed here may be applied to discretise many dissipative stochastic partial differential and difference equations.

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1 Introduction

Introduction

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Stochastic PDEs are an ideal that possess complex dynamics across all space and time scales.[1]

Spatial discretisations transform stochastic PDEs into a finite set of stochastic DEs, then . . . .

Stochastic centre manifold theory supports discretisation:

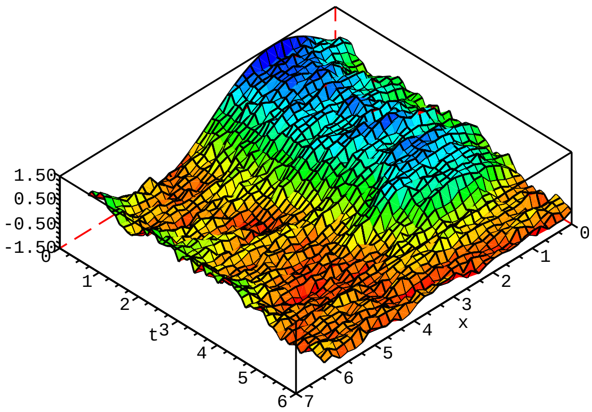
- resolves subgrid microscale processes to close macroscale;
- memory implies one cannot prescribe ‘order parameters’;
- subgrid dynamics correlates noise;
- irreducible quadratic noise generates mean ‘drift’.

1.1 Stochastic reaction-diffusion for example

Introduction Stochastic reaction-diffusion for example

The stochastic reaction-diffusion equation for $u(x,t)$:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \alpha(u - u^3) + \sigma\phi(x,t) \qquad \text{(Stratonovich)}$$



How can we resolve such seething mess on $\mathcal{O}(1)$ lengths and times? Say coarse spatial grid $\Delta x = h = 1.5$.

1.2 Micro SPDE to macro discretisation?

Introduction Micro SPDE to macro discretisation?

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \alpha(u - u^3) + \sigma\phi(x,t)$$

$\downarrow$ method?

$$\begin{aligned}\dot U_j \approx{}& \frac{1}{h^2}(U_{j+1} - 2U_j + U_{j-1}) + \alpha(U_j - U_j^3)\\ &+ \sigma\left[\phi_{j,1} - \tfrac{1}{4}\delta^2\phi_{j,1} - \tfrac{1}{\pi}\mu\delta\phi_{j,2} - \tfrac{1}{4\pi}\delta^2\phi_{j,3}\right]\\ &+ 3\alpha\left[\tfrac{1}{2}U_j(\mu\delta U_j)^2 + \tfrac{1}{4}(\delta^2 U_j)(\mu\delta U_j)^2 + \tfrac{1}{8}U_j(\delta^2 U_j)^2 + \tfrac{1}{48}(\delta^2 U_j)^3\right]\end{aligned}$$

[1] 070214101.mov and MicroMovie/microa.mov (Souganidis)

Even linear diffusion is subtle: set $\alpha = 0$ in above.

Subgrid noise leaks into adjacent elements . . . . but no memory?

2 Avoid memory integrals

Avoid memory integrals

Methods of singular perturbations and averaging avoid memory integrals at the cost of forming a weak model.

Consider stochastic dynamics (slow manifold) of toy SDE [simavs]

$$\dot{x} = -xy \quad \text{and} \quad \dot{y} = -y + x^2 - 2y^2 + \sigma\phi(t)\,.$$

Averaging does not see the noise induced bifurcation ('drift').

Most choose parametrisation $y = h(x,t)$ with *horrible* consequence that slow model has fast memory: $\dot{x} \approx -x^3 - \sigma x e^{-t} \star \phi$. $\left(e^{-t} \star \phi = \int_0^\infty e^{-\tau}\phi(t-\tau)\,d\tau\right)$

A stochastic coordinate tramsform (nothing lost) additionally parametrises slow x in terms of a 'true' slow X [zsimsnf].

Avoid memory integrals

Nothing lost in stochastic coordinate transform[2] (e.g. Arnold 2003)

$$y \approx Y + X^2 + 2Y^2 + \sigma(1+4Y)e^{-t} \star \phi\,,$$
$$x \approx X + XY + \sigma X e^{-t} \star \phi\,,$$

results in dynamics

$$\dot{Y} \approx Y(-1 - 2X^2 - 4\sigma\phi)\,, \qquad (\Rightarrow Y \to 0 \text{ quickly})$$
$$\dot{X} \approx -X^3 - \sigma X \phi\,. \qquad (\Rightarrow X \text{ is 'slow'})$$

All memory integrals go into the location of the stochastic slow manifold, leaving a 'true' slow model in X.

Generalising, no memory need be in SPDE model.

But cost is 'order parameter' $U_j(t) \neq u(X_j,t)$, the field grid value.

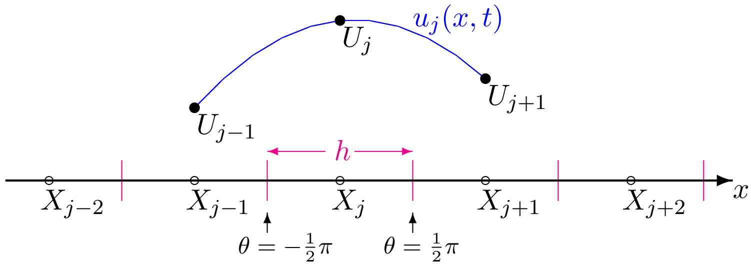
3 Interelement coupling discretises SPDEs

3.1 Divide space into overlapping elements and couple

Interelement coupling discretises SPDEs Divide space into overlapping elements and couple

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \alpha(u - u^3) + \sigma\phi(x,t)$$

[2] http://www.maths.adelaide.edu.au/anthony.roberts/sdenf.html



Divide space and couple elements with homotopy parameter γ:

$$u_j(X_{j\pm1}, t) = \gamma u_{j\pm1}(X_{j\pm1}, t) + (1-\gamma) u_j(X_j, t).$$

Interelement coupling discretises SPDEs Divide space into overlapping elements and couple

Such coupling beautifully consistent . . . . but not self-adjoint.

Self-adjoint coupling Alternatively, replace implied continuity of derivative at mid-point with the derivative condition

$$\begin{aligned} u_{jx}(X_j^+, t) - u_{jx}(X_j^-, t) &= (1-\gamma)\left[u_{jx}(X_{j+1}, t) - u_{jx}(X_{j-1}, t)\right] \\ &\quad - \gamma\left[u_{j+1,x}(X_j, t) - u_{j-1,x}(X_j, t)\right] \end{aligned}$$

Then the linear system on the overlapping elements is self-adjoint.

This alternative may give more immediate theoretical support.

Potential: same approach maps fine lattice dynamics to a model on a coarser lattice . . . . needs research.

3.2 Theoretical support based upon $\alpha = \sigma = \gamma = 0$

Interelement coupling discretises SPDEs Theoretical support based upon $\alpha = \sigma = \gamma = 0$

In the absence of nonlinearity, noise and coupling, solutions decay exponentially quickly to piecewise constant on m elements (say).

$\Rightarrow$ there exists an m-D stochastic slow manifold—Boxler (1989) & Arnold (2003)—in a finite domain of (α, σ, γ) space.

Parametrised by a measure of field in each element, $U_j(t)$.

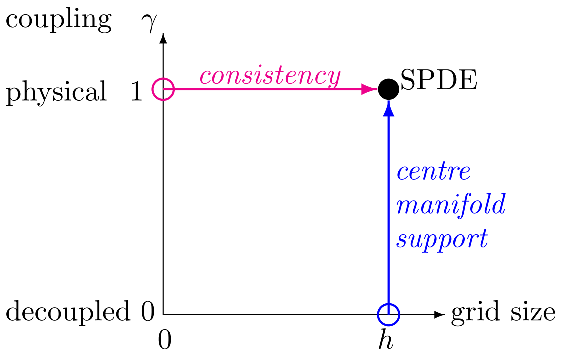
Valid exponentially quickly, for all time . . . . from full transform.

Subgrid fields obtained via pre-solving SPDE in elements.

Although local to $\gamma = 0$; assume the fully coupled case of $\gamma = 1$ is in the finite domain . . . . needs validation.

Theoretical support based upon $\alpha = \sigma = \gamma = 0$

Novel justification of discretisations



3.3 Represent noise in the local basis

In principle we represent the subgrid fields in each element in the local orthogonal basis $(-\pi < \theta < \pi)$:

$$u_j(x,t) = u_{j,1}(t) + u_{j,2}(t)\sin\theta + u_{j,3}(t)\sin 2|\theta| + u_{j,4}(t)\sin 2\theta + u_{j,5}(t)\cos 2\theta + \cdots .$$

Then the SPDE becomes the infinite set of SDEs $\dot u_{j,k} = -\beta_k u_{j,k} + \cdots$ for decay rates $0 = \beta_1 < \beta_2 < \beta_3 < \cdots$.

Thus $u_{j,1}$ are approximately the slow variables, and $u_{j,k}$ for $k \geq 2$ the fast decaying variables. In principle, a near identity stochastic coordinate transform exists, $u_{j,k} \approx U_{j,k}$, that makes $U_{j,k} = 0$ for $k \geq 2$ exponentially quickly attractive, leaving the long term evolution to be parametrised by $U_{j,1}$, or equivalently by the psuedo-grid-values $U_j(t)$.

Interelement coupling discretises SPDEs Represent noise in the local basis

In jth element $(-\pi < \theta < \pi)$ expand noise in basis eigenmodes:

$$\phi(x,t) = \phi_{j,1}(t) + \phi_{j,2}(t)\sin\theta + \phi_{j,3}(t)\sin 2|\theta| + \phi_{j,4}(t)\sin 2\theta + \phi_{j,5}(t)\cos 2\theta + \cdots$$

Need to 'lift' $\phi(x,t)$ into overly-complete, local bases . . . . issues.

Computer algebra constructs subgrid fields (part xform)

$$u_j(x,t) = U_j + \gamma\left[(\theta/\pi)\mu\delta + \tfrac{1}{2}|\theta/h|\delta^2\right]U_j + \sigma\left[\sin\theta\, e^{-\frac{\pi^2}{h^2}t}\star\phi_{j,2} + \sin 2|\theta|\, e^{-4\frac{\pi^2}{h^2}t}\star\phi_{j,3}\right] - \alpha\sigma(1-3U_j^2)\left[\sin\theta\left(e^{-\frac{\pi^2}{h^2}t}\star\right)^2\phi_{j,2} + \sin 2|\theta|\left(e^{-4\frac{\pi^2}{h^2}t}\star\right)^2\phi_{j,3}\right] + \cdots$$

3.4 Full coupling, $\gamma = 1$, gives discrete model

Interelement coupling discretises SPDEs Full coupling, $\gamma = 1$, gives discrete model

The corresponding evolution An 'SPDE' close to original.

$$\begin{aligned}\dot U_j &\approx \gamma^2 \frac{1}{h^2}\delta^2 U_j + \alpha(U_j - U_j^3) + \sigma\phi_{j,1} \\ &\quad - \gamma\sigma\left[\tfrac14\delta^2\phi_{j,1} + \tfrac1\pi\mu\delta\phi_{j,2} + \tfrac{1}{4\pi}\delta^2\phi_{j,3}\right] \\ &\quad + 3\alpha\gamma\left[\tfrac12 U_j(\mu\delta U_j)^2 + \tfrac14(\delta^2U_j)(\mu\delta U_j)^2 + \tfrac18 U_j(\delta^2U_j)^2 + \tfrac{1}{48}(\delta^2U_j)^3\right] \\ &\quad - \alpha\sigma^2h^2U_j\left[\tfrac{3}{2\pi^2}\phi_{j,2}e^{-\frac{\pi^2}{h^2}t}\star\phi_{j,2} + \tfrac{3}{8\pi^2}\phi_{j,3}e^{-4\frac{\pi^2}{h^2}t}\star\phi_{j,3}\right] \\ &\quad + \mathcal{O}\big(\sigma^5 + \alpha^{5/2} + \gamma^{5/2}\big)\,.\end{aligned}$$

Resolving subgrid $\Rightarrow$ multiplicative noise . . . . 'stabilising' $u = 0$.

Curses, the σ^2 term has micro-time memory integrals!

4 Irreducible quadratic memory integrals

4.1 Return to the toy system

Irreducible quadratic memory integrals Return to the toy system

$$\dot x = -xy \quad\text{and}\quad \dot y = -y + x^2 - 2y^2 + \sigma\phi(t)\,.$$

Coordinate transform shows stochastic slow manifold model is

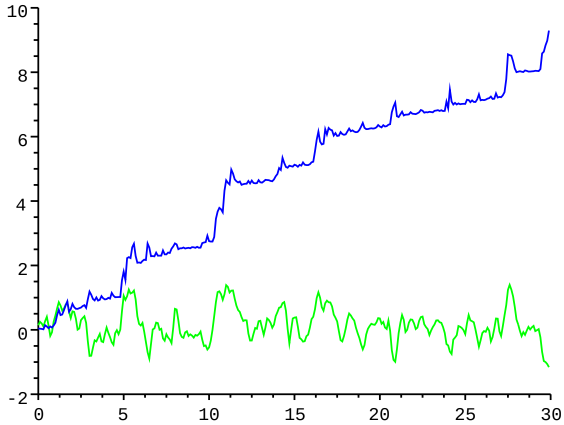
$$\begin{aligned}&x = X + \sigma X e^{-t}\star\phi - \sigma^2 X(2e^{-t}\star - \tfrac12)(e^{-t}\star\phi)^2 + \cdots \\ &y = X^2 + \sigma e^{-t}\star\phi - 2\sigma^2 e^{-t}\star(e^{-t}\star\phi)^2 + \cdots \\ &\text{with evolution} \\ &\dot X = -X^3 - \sigma X\phi + 2\sigma^2 X\phi e^{-t}\star\phi + \cdots\end{aligned}$$

Find terms such as $\phi e^{-t}\star\phi$ are irreducible in this approach—they are their own slow manifold! . . . . must invoke weak models.

4.2 Effective drift and noise $\phi e^{-t}\star\phi \approx \frac12 + \frac{1}{\sqrt2}\psi$

Irreducible quadratic memory integrals Effective drift and noise in $\phi e^{-t}\star\phi \approx \frac12 + \frac{1}{\sqrt2}\psi$

Not only is $E[\text{SSM}] \neq$ deterministic SM, but



$\int \phi e^{-t} \star \phi \, dt \ e^{-t} \star \phi$
$\Rightarrow \phi e^{-t} \star \phi \approx \frac{1}{2} + \frac{1}{\sqrt{2}}\psi$
for independent $\psi(t)$,
algebraically emerges.

4.3 Fokker–Planck slow manifold generates drift and noise

Irreducible quadratic memory integrals Fokker–Planck slow manifold generates drift and noise

The full hierarchy of such quadratic interactions analysed via the deterministic slow manifold of Fokker–Planck equations $\Rightarrow$

$$\begin{aligned}\dot{U}_j \approx{}& \frac{1}{h^2}\delta^2 U_j + \alpha(U_j - U_j^3) \\ &+ \sigma\left[\phi_{j,1} - \left(\tfrac{1}{4}\delta^2\phi_{j,1} - \tfrac{1}{\pi}\mu\delta\phi_{j,2} - \tfrac{1}{4\pi}\delta^2\phi_{j,3}\right)\right] \\ &+ 3\alpha\left[\tfrac{1}{2}U_j(\mu\delta U_j)^2 + \tfrac{1}{4}(\delta^2 U_j)(\mu\delta U_j)^2 + \tfrac{1}{8}U_j(\delta^2 U_j)^2 + \tfrac{1}{48}(\delta^2 U_j)^3\right] \\ &- \alpha\sigma^2 h^2 U_j\left[\frac{15}{16\pi^2} + \frac{3\sqrt{65}h}{16\sqrt{2}\pi^3}\psi(t)\right]\end{aligned}$$

Multitude of quadratic interactions generate

- ‘mean drift’, and
- new noise brought up from the microscale.

Conclusion

- Methodology closes macroscale discretisations — accounts for seething mess of microscale noise.
- Being based upon coordinate transform, there is an asymptotically nearby SPDE to what the model SDEs predict.
- Avoid memory integrals.
- Quadratic noise abstracts drift and effectively new noises from the microscale.
- Computer algebra handles extensive details.[3]
- Lots of open problems.

[3] `http://www.maths.adelaide.edu.au/anthony.roberts/Modelling` introduces many of the basic methods—almost all deterministic as yet.