Complexity Results for S(P)DEs

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I. Complexity of Numerical Problems

1. **Computational problem**: approximation of SPDEs, SDEs, ...
2. **Computational means**: class $\mathcal{A}$ of algorithms.
3. **Quality criterion**: error and cost of an algorithm.
4. **Minimal error and complexity**:

$$e(n) = \inf\{\mathrm{error}(A) : A \in \mathcal{A} \text{ such that } \mathrm{cost}(A) \leq n\},$$

$$\mathrm{comp}(\varepsilon) = \inf\{\mathrm{cost}(A) : A \in \mathcal{A} \text{ such that } \mathrm{error}(A) \leq \varepsilon\}.$$

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Leads to

- benchmarks for existing algorithms,
- definition of optimal algorithms,
- construction of new algorithms (sometimes).

Typical result

$$e(n) \asymp n^{-\alpha},$$

consists of

- **upper bound**: construction of algorithms $A_n \in \mathcal{A}$ such that $\exists\, c > 0 \;\; \forall\, n \in \mathbb{N}$:

$$\mathrm{cost}(A_n) \leq n \quad \wedge \quad \mathrm{error}(A_n) \leq c \cdot n^{-\alpha},$$

- **lower bound**: $\exists\, c > 0 \;\; \forall$ algorithm $A \in \mathcal{A} \;\; \forall\, n \in \mathbb{N}$:

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Monographs

Traub, Wasilkowski, Woźniakowski (1988), Novak (1988),

. . . , Plaskota (1996), . . . , Ritter (2000) . . .

II. Computational Problems for S(P)DEs

SDE

$$dX(t) = A(t, X(t))\,dt + B(t, X(t))\,dW(t)$$

with state space $H = \mathbb{R}^d$ and m-dimensional Brownian motion W.

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SPDE

$$dX(t) = \big(\Delta X(t) + A(t, X(t))\big)\,dt + B(t, X(t))\,dW(t)$$

with infinite-dim. state space H and infinite-dim. Brownian motion W.

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with infinite-dim. state space H and infinite-dim. Brownian motion W.

Solution $X = (X(t))_{t\in[0,T]}$ is a stochastic process in H with

$$X = \Phi(W, A, B)$$

for fixed initial value $X(0) = h \in H$ (and generator Δ).

Computational problems

- Strong approximation: approximate the solution X.
- Weak approximation: approximate the distribution P_X of X.
- Cubature: approximate integrals $E(f(X)) = \int f \, dP_X$ w.r.t. P_X.

Analogously, for the solution X at a single time instance T.

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Remark

- Reasonable, but not mandatory,

  strong approximation $\rightsquigarrow$ weak approximation $\rightsquigarrow$ cubature.

- Key for proving lower bounds: every algorithm may only use partial information about W, A, B, and f.

III. Strong Approximation

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3. Quality criterion:

$$\mathrm{error}(\hat{X}) = \Big(E(\|X - \hat{X}\|^2)\Big)^{1/2}$$

   for a norm $\|\cdot\|$ on $\mathfrak{X} = C([0,1], H)$, say, or

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Sometimes $$\mathrm{cost}(\hat{X}) = E\,(\#\text{oracle calls}).$$

Strong approximation of SDEs

- **upper bounds**:

  extensively studied

- **lower bounds**:

  Clark, Cameron (1980), *Cambanis, Hu (1996)*,

  Hofmann, Müller-Gronbach, R (2000, . . .),

  Müller-Gronbach (2002, . . .), Neuenkirch (2006,. . .), . . .

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Strong approximation of SPDEs

- **upper bounds**:
  Grecksch, Kloeden (1996), Gyöngy, Nualart (1997), . . .
  Jentzen, Kloeden (2010, . . .), Jentzen, Röckner (2010), . . .
- **lower bounds**:
  Davie, Gaines (2001), Müller-Gronbach, R (2007,. . .)

Consider a **scalar SDE**. Assume that

(i) A and B satisfy standard smoothness assumptions,

(ii) the oracles for $G = A$ and $G = B$ provide $G(t,h)$ or $G^{(0,1)}(t,h)$ for any $t \in [0,1]$ and $h \in \mathbb{R}$,

(iii) the oracle for W provides $W(t,\omega)$ for any $t \in [0,1]$.

Theorem 1 *Müller-Gronbach (2002, 2004)*

For approximation at T

$$e(n) \approx c_1(A, B) \cdot n^{-1}.$$

For approximation in $\mathfrak{X} = L_\infty([0, T], \mathbb{R})$

$$e(n) \approx c_2(A, B) \cdot (n/\ln n)^{-1/2}.$$

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Remarks

- Upper bounds via adaptive step-size control; uniform time discretization is suboptimal.
- Results available for systems of SDEs and for L_p-norms.
- Partial results for more powerful oracles.

Consider a **stochastic heat equation**

$$dX(t) = \Delta X(t)\,dt + B(t, X(t))\,dW(t).$$

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(iii)
$$W(t) = \sum_{\boldsymbol{i} \in \mathbb{N}^d} |\boldsymbol{i}|_2^{-\gamma/2} \cdot \beta_{\boldsymbol{i}}(t) \cdot h_{\boldsymbol{i}}$$

with eigenfunctions $h_{\boldsymbol{i}}$ of Δ, independent scalar Bms $(\beta_{\boldsymbol{i}})_{\boldsymbol{i} \in \mathbb{N}^d}$, either $\gamma = 0$ and $d = 1$ or $\gamma > d \in \mathbb{N}$.

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(iv) The oracle for W provides $\beta_{\boldsymbol{i}}(t, \omega)$ for any $\boldsymbol{i} \in \mathbb{N}^d$ and $t \in [0,T]$.

Theorem 2 *Müller-Gronbach, R (2007)*

For approximation in $\mathfrak{X} = L_2([0,T], H)$

$$e(n) \succeq n^{-\alpha},$$

where

$$\alpha = \frac{\min(\gamma - d, d) + 2}{2(d+2)}$$

if $\gamma = 0$ and $d = 1$ or $\gamma \in]d, \infty[\setminus \{2d\}$ and $d \in \mathbb{N}$.

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Lower bounds are sharp, i.e., $e(n) \asymp n^{-\alpha}$, if

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Unknown: Sharp bound in the case (ii) for

$$\mathrm{cost}(\hat{X}) = E\left(\#\text{oracle calls} + \#\text{arith. op's}\right).$$

Remarks

- Upper bounds via non-uniform time discretization of W and hyperbolic cross approximation of X; uniform time discretization is suboptimal.
- For approximation at T, see *Müller-Gronbach, R, Wagner (2007)*, *Henkel (2010)*, *Jentzen, Kloeden (2010, ...), Jentzen, Röckner (2010)*.
- More powerful oracle provides $\xi(\omega)$ for

$$\xi \in \overline{\mathrm{span}}\{\langle W(t), h\rangle : t \in [0, T],\ h \in H\}.$$

See *Davie, Gaines (2001)* for a lower bound for a particular equation with $d = 1$ and $\gamma = 0$.

IV. Weak Approximation

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3. Quality criterion for a ‘random function generator’ $\hat{X}$:

$$\mathrm{error}(\hat{X}) = \rho(P_X, P_{\hat{X}}) \qquad \text{or} \qquad \mathrm{error}(\hat{X}) = \rho(P_{X(T)}, P_{\hat{X}(T)})$$

   for some metric ρ on the space of probability measures on $\mathfrak{X} = C([0,1], H)$ or on H, resp.,

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for some metric ρ on the space of probability measures on $\mathfrak{X} = C([0,1], H)$ or on H, resp., and

$$\mathrm{cost}(\hat{X}) = \mathbb{E}\,\big(\#\text{oracle calls} + \#\text{arithmetical operations} + \#\text{calls of random number generator}\big)\,.$$

Weak approximation of SDEs

- **upper bounds**:

  extensively studied for $P_{X(T)}$

- **lower bounds**:

  Creutzig, Müller-Gronbach, R (2008), *Slassi (2010)*.

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Weak approximation of SPDEs

- **upper bounds**:

  Hausenblas (2003), Shardlow (2003),

  Debussche, Printems (2009), Geissert, Kovácz, Larsson (2009),

  Lindner (2010), . . .

- **lower bounds**:

  —

Remark: Consider the Wasserstein metric

$$\rho(\mu, \widehat{\mu}) = \inf_{\nu} \int_{\mathfrak{X}\times\mathfrak{X}} \|x - y\| \, d\nu(x, y)$$

with $\inf$ over all probability measures ν on $\mathfrak{X} \times \mathfrak{X}$ with marginals $\mu, \widehat{\mu}$.

By the Kantorovich-Rubinstein Theorem,

$$\rho(\mu, \widehat{\mu}) = \sup_{f \in \mathrm{Lip}(1)} \left| \int_{\mathfrak{X}} f \, d\mu - \int_{\mathfrak{X}} f \, d\widehat{\mu} \right| .$$

Consider a **scalar SDE**.

Goal: approximate P_X by means of a 'random function generator' $\hat{X}$ with $P_{\hat{X}}$ supported on the space

$$\{x \in L_\infty([0,1]) : x \text{ piecewise linear}\}$$

of linear splines with free knots.

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Theorem 3 *Creutzig, Müller-Gronbach, R (2007), Slassi (2010)*

For $\mathfrak{X} = L_\infty([0,1], \mathbb{R})$ and the Wasserstein metric ρ

$$e(n) \asymp n^{-1/2}.$$

For equations with additive noise

$$e(n) \approx c(A, B) \cdot n^{-1/2}.$$

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Remarks

- The same asymptotics holds P-a.s.
- For nonlinear approximation of stochastic processes, see also *Cohen, d'Ales (1997), Kon, Plaskota (2005), Dahlke et al. (2010)*.

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Remarks

- For deterministic approximation of P_X by discrete measures (quantization), $e(n) \asymp (\ln n)^{-1/2}$, see *Creutzig, Dereich, Müller-Gronbach, R (2009), Müller-Gronbach, R (2010)*.
- For deterministic approximation of $P_{X(T)}$ by discrete measures, see *Müller-Gronbach, R, Yaroslavtseva (2010)*.

V. Cubature

1. Computational problem: approximate $E(f(X))$ for $f : \mathfrak{X} \to \mathbb{R}$.
2. Computational means: real number model, oracle for A, B, and f, and a random number generator.

   Choose any scale of finite-dim. subspaces $\mathfrak{X}_1 \subset \mathfrak{X}_2 \subset \ldots \subset \mathfrak{X}$. The oracle for f returns $f(x)$ for

$$x \in \bigcup_{m=1}^{\infty} \mathfrak{X}_m.$$

   The cost per call is $\inf\{\dim \mathfrak{X}_m : x \in \mathfrak{X}_m\}$.

3. Quality criterion:

$$\mathrm{error}(\hat{X}) = \sup_{f \in F} \Big(\mathbb{E} \big| E(f(X)) - \hat{X}(f) \big|^2 \Big)^{1/2}.$$

and

$$\mathrm{cost}(\hat{X}) = \sup_{f \in F} \mathbb{E} \big(\text{cost for oracle calls} + \#\text{arithmetical operations} + \#\text{calls of random number generator}\big).$$

Consider an **SDE**, where $\mathfrak{X} = C([0,1], \mathbb{R}^d)$, and $F = \mathrm{Lip}(1)$, i.e.,

$$|f(x) - f(y)| \leq \|x - y\|_{\mathfrak{X}}, \qquad x, y \in \mathfrak{X}.$$

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Theorem 4 *Creutzig, Dereich, Müller-Gronbach, R (2009)*

$$n^{-1/2} \preceq e(n) \preceq n^{-1/2} \cdot \log n.$$

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Remark

- Upper bound via multi-level algorithm.

  See *Heinrich (1998, . . .), Giles (2008,. . .)*

- Minimal errors of deterministic algorithms $(\log n)^{-1/2}$.

- Integration on the sequence space $\mathbb{R}^{\mathbb{N}}$, see

  Hickernell, Wang (2002), Niu, Hickernell (2009), . . . ,

  Plaskota, Wasilkowski (2010), . . .