Taylor Expansions and Numerical Approximations for Stochastic Partial Differential Equations

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1. Taylor expansions for SODEs
2. Taylor expansions for SPDEs
3. A new numerical method for SPDEs with non-additive noise
4. A new numerical method for SPDEs with additive noise
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4. A new numerical method for SPDEs with additive noise
Let $T > 0$ and let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $f, g : \mathbb{R} \to \mathbb{R}$ be smooth functions and let $(W_t)_{t \in [0, T]}$ be a scalar Brownian motion.

Consider the SODE:

$$dX_t = f(X_t) \, dt + g(X_t) \, dW_t,$$

which is understood as

$$X_t = X_0 + \int_0^t f(X_s) \, ds + \int_0^t g(X_s) \, dW_s$$

$\mathbb{P}$-a.s. for all $t \in [0, T]$. Applying Itô’s formula to the integrands above yields

$$X_t \approx X_0 + f(X_0) \cdot t + g(X_0) \cdot W_t + \frac{1}{2} \cdot g'(X_0) \cdot g(X_0) \cdot \left( (W_t)^2 - t \right) \quad \mathbb{P}\text{-a.s.}$$

Milstein’s approximation (1974).
Let $T > 0$ and let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $f, g : \mathbb{R} \to \mathbb{R}$ be smooth functions and let $(W_t)_{t \in [0,T]}$ be a scalar Brownian motion. Consider the SODE:

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\frac{dX_t}{dt} = f(X_t) \frac{dt}{dt} + g(X_t) \frac{dW_t}{dW_t},
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The corresponding numerical scheme, the so-called **Milstein scheme** (or also Taylor scheme of (strong) order 1.0), is then given by \( Y_N^0 = X_0 \) and

\[
Y_{n+1}^N = Y_n^N + f(Y_n^N) \cdot \frac{T}{N} + g(Y_n^N) \cdot \left( W_{(n+1)T}^N - W_{nT}^N \right) \]
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for every \( n \in \{0, 1, \ldots, N - 1\} \), \( N \in \mathbb{N} \) and is of (strong) order 1.

It is very easy to simulate and impressively efficient for one-dimensional SODEs in comparison to e.g. Euler’s method which is of (strong) order \( \frac{1}{2} \).
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The general theory of Taylor expansions for SODEs and the numerical schemes based on them can be found in the monographs

- Kloeden & Platen (1992)
- Milstein (1995)

(Stochastic Taylor expansions).

Essential constituents:

\[ X_t \approx X_0 + f(X_0) \cdot \int_0^t ds + g(X_0) \cdot \int_0^t dW_s + g'(X_0) g(X_0) \cdot \int_0^t \int_0^s dW_u dW_s \]

Method for deriving Taylor expansions:

- Iterated application of the stochastic fundamental theorem of calculus, i.e. Itô’s formula.
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Itô’s formula for SPDEs? In general not a semi-martingale

Let \((H, \| \cdot \|)\) be a reasonable state space of a SPDE.

- Krylov & Rozovskii (1979): Itô formula for \(F : H \to \mathbb{R}, F(v) = \|v\|^2\), \(v \in H\), with powerful consequences to the variational approach, see also Gyöngy & Krylov (1981) and Prévot & Röckner (2007)
- Gradinaru, Nourdin & Tindel (2005): Itô formula for \(F : H \to \mathbb{R}\) smooth
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For the Taylor expansion approach for SODEs, one would need an Itô formula for \(F : H \to H\) smooth.

If the solution of the SPDE is spatially smooth and the solution is a semi-martingale, then Taylor expansions & Taylor schemes for SPDEs in

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Fix $T > 0$, a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with normal filtration $(\mathcal{F}_t)_{t \in [0, T]}$ and two $\mathbb{R}$-Hilbert spaces $H$ and $U$.

(A1) Let $A : D(A) \subset H \to H$ be a bijective linear operator with negative compact inverse.

(A2) Let $W : [0, T] \times \Omega \to U$ be a standard $Q$-Wiener process with covariance operator $Q : U \to U$. Let $U_0 := Q^{\frac{1}{2}}(U)$.

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\[ dX_t = \left[ AX_t + F(X_t) \right] dt + B(X_t) dW_t, \quad X_0 = \xi \]

for \( t \in [0, T] \).

SPDE in the mild form

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Let $d \in \mathbb{N}$ and consider the SPDE

$$dX_t(x) = \left[ \kappa \Delta X_t(x) + f(x, X_t(x)) \right] dt + b(x, X_t(x)) \, dW_t(x)$$

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$\mathbb{P}$-a.s. for all $t \in [0, T]$. Omitting first and second summand yields the first simple Taylor approximation for SPDEs

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\[ \Delta X_t \approx (e^{At} - I) X_0 + \int_0^t e^{A(t-s)} B(X_0) \, dW_s, \quad \text{i.e.} \]

\[ X_t \approx e^{At} X_0 + \int_0^t e^{A(t-s)} B(X_0) \, dW_s \]
Consider

\[ \Delta X_t = \left( e^{At} - I \right) X_0 + \int_0^t e^{A(t-s)} F(X_s) \, ds + \int_0^t e^{A(t-s)} B(X_s) \, dW_s \]

\(\mathbb{P}\)-a.s. for all \( t \in [0, T] \). Omitting first and second summand yields the first simple Taylor approximation for SPDEs

\[ \Delta X_t \approx \int_0^t e^{A(t-s)} B(X_0) \, dW_s, \quad \text{i.e.} \]

\[ X_t \approx X_0 + \int_0^t e^{A(t-s)} B(X_0) \, dW_s \]

Omitting second summand gives

\[ \Delta X_t \approx \left( e^{At} - I \right) X_0 + \int_0^t e^{A(t-s)} B(X_0) \, dW_s, \quad \text{i.e.} \]

\[ X_t \approx e^{At} X_0 + \int_0^t e^{A(t-s)} B(X_0) \, dW_s \]
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$$\Delta X_t = (e^{At} - I) X_0 + \int_0^t e^{A(t-s)} F(X_s) \, ds + \int_0^t e^{A(t-s)} B(X_s) \, dW_s$$

$\mathbb{P}$-a.s. for all $t \in [0, T]$. Classical Taylor approximations $F(X_s) \approx F(X_0)$ and $B(X_s) \approx B(X_0)$ yield

$$\Delta X_t \approx (e^{At} - I) X_0 + \int_0^t e^{A(t-s)} F(X_0) \, ds + \int_0^t e^{A(t-s)} B(X_0) \, dW_s$$
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\[ = \left( \int_0^t e^{A(t-s)} \, ds \right) F(X_0) \]
Consider

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\[ \mathbb{P}\text{-a.s. for all } t \in [0, T]. \]

Classical Taylor approximations \( F(X_s) \approx F(X_0) \) and \( B(X_s) \approx B(X_0) \) yield

\[ \Delta X_t \approx (e^{At} - I) X_0 + \int_0^t e^{A(t-s)} F(X_0) \, ds + \int_0^t e^{A(t-s)} B(X_0) \, dW_s, \text{ i.e.} \]

\[ \begin{align*}
= & (\int_0^t e^{A(t-s)} \, ds) F(X_0) \\
= & A^{-1} (e^{At} - I) F(X_0)
\end{align*} \]

\[ X_t \approx e^{At} X_0 + A^{-1} (e^{At} - I) F(X_0) + \int_0^t e^{A(t-s)} B(X_0) \, dW_s \]

Exponential Euler approximation (see the final section for more details).
Consider

\[ \Delta X_t = \left( e^{At} - I \right) X_0 + \int_0^t e^{A(t-s)} F(X_s) \, ds + \int_0^t e^{A(t-s)} B(X_s) \, dW_s \]

\( \mathbb{P} \)-a.s. for all \( t \in [0, T] \). Classical Taylor approximations \( F(X_s) \approx F(X_0) \) and \( B(X_s) \approx B(X_0) \) yield

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\[ X_t \approx e^{At} X_0 + A^{-1} (e^{At} - I) F(X_0) + \int_0^t e^{A(t-s)} B(X_0) \, dW_s \]

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= A^{-1} (e^{At} - I) F(X_0)
\]

\[ X_t \approx e^{At} X_0 + A^{-1} (e^{At} - I) F(X_0) + \int_0^t e^{A(t-s)} B(X_0) \, dW_s \]

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\( \mathbb{P} \)-a.s. for all \( t \in [0, T] \). Using \( B(X_s) \approx B(X_0) + B'(X_0) \Delta X_s \) shows

\[ \Delta X_t \approx \left( e^{At} - I \right) X_0 + \int_0^t e^{A(t-s)} F(X_0) \, ds + \int_0^t e^{A(t-s)} B(X_0) \, dW_s \]

\[ + \int_0^t e^{A(t-s)} B'(X_0) \Delta X_s \, dW_s \]

Using \( \Delta X_s \approx \int_0^s e^{A(s-u)} B(X_0) \, dW_u \) (first simple Taylor approximation) recursively yields

\[ \Delta X_t \approx \left( e^{At} - I \right) X_0 + \int_0^t e^{A(t-s)} F(X_0) \, ds + \int_0^t e^{A(t-s)} B(X_0) \, dW_s \]

\[ + \int_0^t e^{A(t-s)} B'(X_0) \left( \int_0^s e^{A(s-u)} B(X_0) \, dW_u \right) \, dW_s \]
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i.e.
$$

$$
X_t \approx e^{At} X_0 + \int_0^t e^{A(t-s)} F(X_0) \, ds + \int_0^t e^{A(t-s)} B(X_0) \, dW_s \\
+ \int_0^t e^{A(t-s)} B'(X_0) \left( \int_0^s e^{A(s-u)} B(X_0) \, dW_u \right) \, dW_s
$$

Infinite dimensional analog of Milstein’s approximation (see the next section of this talk)
Using $\Delta X_s \approx \int_0^s e^{A(s-u)} B(X_0) \, dW_u$ (first simple Taylor approximation) recursively yields

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$$

i.e.

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Infinite dimensional analog of Milstein’s approximation (see the next section of this talk)
Using $\Delta X_s \approx \int_0^s e^{A(s-u)} B(X_0) \, dW_u$ (first simple Taylor approximation) recursively yields

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+ \int_0^t e^{A(t-s)} B'(X_0) \left( \int_0^s e^{A(s-u)} B(X_0) \, dW_u \right) \, dW_s, \quad \text{i.e.}
$$

$$
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$$

Infinite dimensional analog of Milstein’s approximation (see the next section of this talk)
Iterating this idea yields more Taylor approximations such as

\[
X_t \approx e^{At}X_0 + \int_0^t e^{A(t-s)}F(X_0) \, ds + \int_0^t e^{A(t-s)}B(X_0) \, dW_s \\
+ \int_0^t e^{A(t-s)}B'(X_0) \left( e^{As} - I \right) X_0 \, dW_s \\
+ \int_0^t e^{A(t-s)}B'(X_0) \left( \int_0^s e^{A(s-u)}B(X_0) \, dW_u \right) \, dW_s \\
+ \frac{1}{2} \int_0^t e^{A(t-s)}B''(X_0) \left( \int_0^s e^{A(s-u)}B(X_0) \, dW_u, \int_0^s e^{A(s-u)}B(X_0) \, dW_u \right) \, dW_s
\]
Some remarks:

- **References:**
  - “Taylor expansions of solutions of stochastic partial differential equations” (J; DCDS B 2010)

- Systematic theory with appropriate integral operators

- Precisely description of the remainder terms & arbitrarily high orders

- Essential constituents: \[ \int_{0}^{t} e^{A(t-s)} B'(X_0) \left( \int_{0}^{s} e^{A(s-u)} B(X_0) \, dW_u \right) \, dW_s \]

- Method: classical Taylor expansions in Banach spaces & recursion technique

- These Taylor Expansions for SPDEs generalize the Taylor Expansions for ODEs and SODEs.
Some remarks:

- **References:**
  - “Taylor expansions of solutions of stochastic partial differential equations” (J; DCDS B 2010)

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- These Taylor Expansions for SPDEs generalize the Taylor Expansions for ODEs and SODEs.
Content

1. Taylor expansions for SODEs
2. Taylor expansions for SPDEs
3. A new numerical method for SPDEs with non-additive noise
4. A new numerical method for SPDEs with additive noise
Reconsider the SPDE and the infinite dimensional analog of Milstein’s approximation

Reconsider the SPDE

\[ dX_t = \left[ AX_t + F(X_t) \right] dt + B(X_t) dW_t, \quad X_0 = \xi \]

for \( t \in [0, T] \). SPDE in the mild form

\[ X_t = e^{At}X_0 + \int_0^t e^{A(t-s)}F(X_s) \, ds + \int_0^t e^{A(t-s)}B(X_s) \, dW_s \]

\( \mathbb{P} \)-a.s. for all \( t \in [0, T] \). Infinite dimensional analog of Milstein’s approximation:

\[ X_t \approx e^{At}X_0 + \int_0^t e^{A(t-s)}F(X_0) \, ds + \int_0^t e^{A(t-s)}B(X_0) \, dW_s \]

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\[ dX_t = \left[ AX_t + F(X_t) \right] dt + B(X_t) \, dW_t, \quad X_0 = \xi \]

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\[ X_t = e^{At} X_0 + \int_0^t e^{A(t-s)} F(X_s) \, ds + \int_0^t e^{A(t-s)} B(X_s) \, dW_s \]

\( \mathbb{P} \)-a.s. for all \( t \in [0, T] \). Infinite dimensional analog of Milstein’s approximation:

\[ X_t \approx e^{At} X_0 + \int_0^t e^{A(t-s)} F(X_0) \, ds + \int_0^t e^{A(t-s)} B(X_0) \, dW_s \]

\[ + \int_0^t e^{A(t-s)} B'(X_0) \left( \int_0^s e^{A(s-u)} B(X_0) \, dW_u \right) \, dW_s \]
Reconsider the SPDE and the infinite dimensional analog of Milstein’s approximation

Reconsider the SPDE

\[ dX_t = \left[ AX_t + F(X_t) \right] dt + B(X_t) \, dW_t, \quad X_0 = \xi \]

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Let $d = 1$ and consider the SPDE

$$dX_t(x) = \left[ \kappa \frac{\partial^2}{\partial x^2} X_t(x) + f(x, X_t(x)) \right] dt + b(x, X_t(x)) \, dW_t(x)$$

with $X_t(0) = X_t(1) = 0$ and $X_0(x) = \xi(x)$ for $x \in (0, 1)$ and $t \in [0, T]$ with the covariance operator $Q : H \to H$ given by

$$(Qv)(x) = \sum_{j=1}^{\infty} \frac{2}{j^3} \cos(j\pi x) \int_0^1 \cos(j\pi y) \, v(y) \, dy$$

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A one-dimensional example SPDE

Let $d = 1$ and consider the SPDE

$$dX_t(x) = \left[ \kappa \frac{\partial^2}{\partial x^2} X_t(x) + f(x, X_t(x)) \right] dt + b(x, X_t(x)) \, dW_t(x)$$

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for all $x \in (0, 1)$ and all $v \in H$. 
Goal: Solve the strong approximation problem, i.e. compute \( Y : \Omega \rightarrow H \) such that

\[
\left( \mathbb{E} \left[ \int_0^1 \left| X_T(x) - Y(x) \right|^2 \, dx \right] \right)^{\frac{1}{2}} < \varepsilon
\]

holds for a given precision \( \varepsilon > 0 \) with the least possible computational effort.

Spectral Galerkin approximations: \( P_N : H \rightarrow H, N \in \mathbb{N}, \) given by

\[
(P_N(v))(x) = \sum_{n=1}^N 2 \sin(n\pi x) \int_0^1 \sin(n\pi y) v(y) \, dy
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**Taylor expansions for SODEs**

**Taylor expansions for SPDEs**

A new numerical method for SPDEs with non-additive noise

A new numerical method for SPDEs with additive noise
Goal: Solve the strong approximation problem, i.e. compute $Y : \Omega \rightarrow H$ such that

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Linear implicit Euler scheme and spectral Galerkin approximations

\[ Z_N^n : \Omega \rightarrow P_N(H), \; n \in \{0, 1, \ldots, N^4\}, \; N \in \mathbb{N}, \text{ given by } Z_0^N = P_N(\xi) \text{ and} \]

\[ Z_{n+1}^N = P_N \left( I - \frac{T}{N^4} A \right)^{-1} \left( Z_n^N + \frac{T}{N^4} \cdot f(\cdot, Z_n^N) + b(\cdot, Z_n^N) \cdot \left( W_{(n+1)T}^{N^4} - W_{nT}^{N^4} \right) \right) \]

for all \( n \in \{0, 1, \ldots, N^4 - 1\}, \; N \in \mathbb{N}. \)

- \( N^4 \) time steps are used in \( (Z_n^N)_{n \in \{0,1,\ldots,N^4\}} \)
- \( Z_n^N \in P_N(H) \) for all \( n \in \{0, 1, \ldots, N^4\} \) and \( P_N(H) \) is \( N \)-dimensional
- \( \underbrace{N^4}_{\text{time steps}} \cdot \underbrace{N}_{\text{dim}(P_N(H))} = N^5 \) independent standard normal random variables needed to simulate \( Z_{N^4}^N \approx X_T \)
Linear implicit Euler scheme and spectral Galerkin approximations

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Theorem (e.g. Hausenblas, 2003)

There exist $C_r > 0$, $r \in (0, 2)$, such that

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\left( \mathbb{E} \left[ \int_0^1 |X_T(x) - Z_N^N(x)|^2 \, dx \right] \right)^{1/2} \leq C_r \cdot N^{(r-2)}
$$

holds for all $N \in \mathbb{N}$ and all arbitrarily small $r \in (0, 2)$.

- $Z_N^N$ converges to $X_T$ with order 2
- $N^5$ random variables are needed to simulate $Z_N^N$
- **Conclusion**: about $O(\varepsilon^{-5/2})$ random variables are needed to achieve the desired precision $\varepsilon > 0$
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Taylor expansions for SODEs

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A new numerical method for SPDEs with non-additive noise

A new numerical method for SPDEs with additive noise

Linear implicit Euler scheme and spectral Galerkin approximations

Theorem (e.g. Hausenblas, 2003)

There exist $C_r > 0$, $r \in (0, 2)$, such that

$$
\left( \mathbb{E} \left[ \int_0^1 \left| X_T(x) - Z_{N^4}^N(x) \right|^2 \, dx \right] \right)^{\frac{1}{2}} \leq C_r \cdot N^{(r-2)}
$$

holds for all $N \in \mathbb{N}$ and all arbitrarily small $r \in (0, 2)$.

- $Z_{N^4}^N$ converges to $X_T$ with order 2
- $N^5$ random variables are needed to simulate $Z_{N^4}^N$

Conclusion: about $O(\varepsilon^{-\frac{5}{2}})$ random variables are needed to achieve the desired precision $\varepsilon > 0$
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- $Z_{N^4}^N$ converges to $X_T$ with order 2—
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- **Conclusion:** about $O(\varepsilon^{-\frac{5}{2}})$ random variables are needed to achieve the desired precision $\varepsilon > 0$
Approximating the semigroup yields

\[ X_t \approx e^{At} \left( X_0 + t \cdot F(X_0) + \int_0^t B(X_0) \, dW_s \right) + \int_0^t B'(X_0) \left( \int_0^s B(X_0) \, dW_u \right) \, dW_s \]
A new algorithm for SPDEs with non-additive noise

Reconsider the infinite dimensional analog of Milstein’s approximation

\[ X_t \approx e^{At} X_0 + \int_0^t e^{A(t-s)} F(X_0) \, ds + \int_0^t e^{A(t-s)} B(X_0) \, dW_s \]

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This indicates the numerical method \( Y^N_n : \Omega \rightarrow P_N(H), \)
\( n \in \{0, 1, \ldots, N^2\}, N \in \mathbb{N}, \) given by \( Y^N_0 = \xi \) and

\[ Y^N_{n+1} = P_N e^{A \frac{T}{N^2}} \left( Y^N_n + \frac{T}{N^2} \cdot F(Y^N_n) + B(Y^N_n) \left( \frac{W^N_{(n+1)T}}{N^2} - \frac{W^N_{nT}}{N^2} \right) \right. \]

\[ + \int_{\frac{nT}{N^2}}^{\frac{(n+1)T}{N^2}} B'(Y^N_n) \left( \int_{\frac{nT}{N^2}}^{s} B(Y^N_n) \, dW^N_u \right) \, dW^N_s \]

\( \mathbb{P} \)-a.s. for all \( n \in \{0, 1, \ldots, N^2 - 1\}, N \in \mathbb{N}. \)
Approximating the semigroup yields

\[ X_t \approx e^{A t} \left( X_0 + t \cdot F(X_0) + \int_0^t B(X_0) \, dW_s + \int_0^t B'(X_0) \left( \int_0^s B(X_0) \, dW_u \right) \, dW_s \right) \]

This indicates the numerical method \( Y_N^n : \Omega \rightarrow P_N(H) \), \( n \in \{0, 1, \ldots, N^2\}, N \in \mathbb{N} \), given by \( Y_N^0 = \xi \) and

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Y_N^{n+1} = P_N e^{A \frac{T}{N^2}} \left( Y_N^n + \frac{T}{N^2} \cdot F(Y_N^n) + B(Y_N^n) \left( \frac{W_N^{(n+1)T} - W_N^{nT}}{N^2} \right) \right. \\
+ \int_{\frac{nT}{N^2}}^{\frac{(n+1)T}{N^2}} B'(Y_N^n) \left( \int_{\frac{nT}{N^2}}^{s} B(Y_N^n) \, dW_u^N \right) \, dW_s^N \bigg)
\]

\( \mathbb{P} \)-a.s. for all \( n \in \{0, 1, \ldots, N^2 - 1\}, N \in \mathbb{N} \).
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Y^N_{n+1} = P_N e^{A^T N^2} \left( Y^N_n + \frac{T}{N^2} \cdot F(Y^N_n) + B(Y^N_n) \left( W^N_{(n+1) T} - W^N_{n T} \right) \right) \\
+ \int_{\frac{n T}{N^2}}^{\frac{(n+1) T}{N^2}} B'(Y^N_n) \left( \int_{\frac{n T}{N^2}}^s B(Y^N_n) \ dW^N_u \right) \ dW^N_s
\]

\( \mathbb{P} \)-a.s. for all \( n \in \{0, 1, \ldots, N^2 - 1\}, \ N \in \mathbb{N}. \) A key observation is

\[
\int_{\frac{n T}{N^2}}^{\frac{(n+1) T}{N^2}} B'(Y^N_n) \left( \int_{\frac{n T}{N^2}}^s B(Y^N_n) \ dW^N_u \right) \ dW^N_s \\
= \frac{1}{2} \left( \frac{\partial}{\partial y} b \right)(\cdot, Y^N_n) \cdot b(\cdot, Y^N_n) \cdot \left( \left( W^N_{(n+1) T} - W^N_{n T} \right)^2 - \frac{T}{N^2} \sum_{j=1}^N \mu_j(g_j)^2 \right)
\]

\( \mathbb{P} \)-a.s. for all \( n \in \{0, 1, \ldots, N^2 - 1\}, \ N \in \mathbb{N} \) where \( (\mu_j)_{j \in \mathbb{N}} \) and \( (g_j)_{j \in \mathbb{N}} \) are the eigenvalues and eigenfunctions of \( Q \) respectively.
\[ Y^N_n : \Omega \rightarrow P_N(H), \quad n \in \{0, 1, \ldots, N^2\}, \quad N \in \mathbb{N}, \text{ given by } Y^N_0 = P_N(\xi) \text{ and } \]

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+ \left. \int_{\frac{nT}{N^2}}^{\frac{(n+1)T}{N^2}} B'(Y^N_n) \left( \int_{\frac{nT}{N^2}}^{s} B(Y^N_n) \, dW^N_u \right) \, dW^N_s \right) \]

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\[ = \frac{1}{2} \left( \frac{\partial}{\partial y} b \right)(\cdot, Y^N_n) \cdot b(\cdot, Y^N_n) \cdot \left( \left( W^N_{(n+1)} - W^N_{nT} \right) \right)^2 - \frac{T}{N^2} \sum_{j=1}^{N} \mu_j(g_j)^2 \]

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Y^N_{n+1} = P_N e^{A^{\frac{T}{N^2}}} \left( Y^N_n + \frac{T}{N^2} \cdot F(Y^N_n) + B(Y^N_n) \left( W^N_{(n+1)\frac{T}{N^2}} - W^N_{n\frac{T}{N^2}} \right) \right)

+ \int_{\frac{nT}{N^2}}^{\frac{(n+1)T}{N^2}} B'(Y^N_n) \left( \int_{\frac{nT}{N^2}}^{s} B(Y^N_n) \, dW^N_u \right) \, dW^N_s
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= \frac{1}{2} \left( \frac{\partial}{\partial y} b \right)(\cdot, Y^N_n) \cdot b(\cdot, Y^N_n) \cdot \left( \left( W^N_{(n+1)\frac{T}{N^2}} - W^N_{n\frac{T}{N^2}} \right)^2 - \frac{T}{N^2} \sum_{j=1}^{N} \mu_j(g_j)^2 \right)
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$Y_N^n : \Omega \rightarrow P_N(H)$, $n \in \{0, 1, \ldots, N^2\}$, $N \in \mathbb{N}$, given by $Y_N^0 = P_N(\xi)$ and

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Y_N^{n+1} = P_N e^{A_T N^{-2}} \left( Y_N^n + \frac{T}{N^2} \cdot f(\cdot, Y_N^n) + b(\cdot, Y_N^n) \cdot \left( \frac{W_{(n+1)T}^N}{N^2} - \frac{W_{nT}^N}{N^2} \right) \right)
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$$
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\]

$$
= \frac{1}{2} \left( \frac{\partial}{\partial y} b \right)(\cdot, Y_N^n) \cdot b(\cdot, Y_N^n) \cdot \left( \left( \frac{W_{(n+1)T}^N}{N^2} - \frac{W_{nT}^N}{N^2} \right)^2 - \frac{T}{N^2} \sum_{j=1}^N \mu_j (g_j)^2 \right)
$$

$\mathbb{P}$-a.s. for all $n \in \{0, 1, \ldots, N^2 - 1\}$, $N \in \mathbb{N}$ where $(\mu_j)_{j \in \mathbb{N}}$ and $(g_j)_{j \in \mathbb{N}}$ are the eigenvalues and eigenfunctions of $Q$ respectively.
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\( Y^N_n : \Omega \to P_N(H), \quad n \in \{0, 1, \ldots, N^2\}, \quad N \in \mathbb{N}, \) given by \( Y^N_0 = P_N(\xi) \) and

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Y^N_{n+1} = P_N e^{A \frac{T}{N^2}} Y^N_n + \frac{T}{N^2} \cdot f(\cdot, Y^N_n) + b(\cdot, Y^N_n) \cdot \left( \frac{W_{(n+1)T}}{N^2} - \frac{W_{nT}}{N^2} \right) \\
+ \frac{1}{2} \left( \frac{\partial}{\partial y} b \right)(\cdot, Y^N_n) \cdot b(\cdot, Y^N_n) \cdot \left( \left( \frac{W_{(n+1)T}}{N^2} - \frac{W_{nT}}{N^2} \right)^2 - \frac{T}{N^2} \sum_{i=1}^{N} \mu_i(g_i)^2 \right)
\]

for all \( n \in \{0, 1, \ldots, N^2 - 1\}, \quad N \in \mathbb{N}. \)

- \( N^2 \) time steps are used in \( (Y^N_n)_{n \in \{0, 1, \ldots, N^2\}} \)
- \( Y^N_n \in P_N(H) \) for all \( n \in \{0, 1, \ldots, N^2\} \) and \( P_N(H) \) is \( N \)-dimensional
- \( \frac{N^2 \text{ time steps}}{\text{dim}(P_N(H))} = N^3 \) independent standard normal random variables needed to simulate \( Y^N_{N^2} \approx X_T \)
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\[ Y^N_{n+1} = P_N e^{AT_{N^2}} \left( Y^N_n + \frac{T}{N^2} \cdot f(\cdot, Y^N_n) + b(\cdot, Y^N_n) \cdot \left( \frac{W^{N(n+1)T}_{nT}}{N^2} - \frac{W^N_{nT}}{N^2} \right) \right) + \frac{1}{2} \left( \frac{\partial b}{\partial y} \right)(\cdot, Y^N_n) \cdot b(\cdot, Y^N_n) \cdot \left( \left( \frac{W^{N(n+1)T}_{nT}}{N^2} - \frac{W^N_{nT}}{N^2} \right)^2 - \frac{T}{N^2} \sum_{i=1}^{N} \mu_i(g_i)^2 \right) \]

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Y^{n+1}_N = P_N e^{At \frac{T}{N^2}} \left( Y_n^N + \frac{T}{N^2} \cdot f(\cdot, Y^N_n) + b(\cdot, Y^N_n) \cdot \left( W_{(n+1)T}^N - W_{nT}^N \frac{T}{N^2} \right) \right) \\
+ \frac{1}{2} \left( \frac{\partial}{\partial y} b \right)(\cdot, Y^N_n) \cdot b(\cdot, Y^N_n) \cdot \left( \left( W_{(n+1)T}^N - W_{nT}^N \frac{T}{N^2} \right)^2 - \frac{T}{N^2} \sum_{i=1}^N \mu_i (g_i)^2 \right)
\]

for all $n \in \{0, 1, \ldots, N^2 - 1\}$, $N \in \mathbb{N}$.

- $N^2$ time steps are used in $(Y^n_N)_{n \in \{0, 1, \ldots, N^2\}}$
- $Y^n_N \in P_N(H)$ for all $n \in \{0, 1, \ldots, N^2\}$ and $P_N(H)$ is $N$-dimensional

\[
\underbrace{N^2}_{\text{time steps}} \cdot \underbrace{N}_{\text{dim}(P_N(H))} = N^3 \quad \text{independent standard normal random variables needed to simulate } Y^N_{N^2} \approx X_T
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\]

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- \[
\underbrace{N^2}_{\text{time steps}} \cdot \underbrace{N}_{\text{dim}(P_N(H))} = N^3
\]

Independent standard normal random variables needed to simulate \( Y^N_{N^2} \approx X_T \)
Taylor expansions for SODEs

Taylor expansions for SPDEs

A new numerical method for SPDEs with non-additive noise

A new numerical method for SPDEs with additive noise

$Y^N_n : \Omega \rightarrow P_N(H)$, $n \in \{0, 1, \ldots, N^2\}$, $N \in \mathbb{N}$, given by $Y^N_0 = P_N(\xi)$ and

$Y^N_{n+1} = P_N e^{A T N^2} \left( Y^N_n + \frac{T}{N^2} \cdot f(\cdot, Y^N_n) + b(\cdot, Y^N_n) \cdot \left( \frac{W^N_{(n+1)T}}{N^2} - \frac{W^N_{nT}}{N^2} \right) \right.$

$+ \frac{1}{2} \left( \frac{\partial}{\partial y} b \right)(\cdot, Y^N_n) \cdot b(\cdot, Y^N_n) \cdot \left( \left( \frac{W^N_{(n+1)T}}{N^2} - \frac{W^N_{nT}}{N^2} \right)^2 - \frac{T}{N^2} \sum_{i=1}^{N} \mu_i(g_i)^2 \right) \bigg) \bigg)$

for all $n \in \{0, 1, \ldots, N^2 - 1\}$, $N \in \mathbb{N}$.

- $N^2$ time steps are used in $(Y^N_n)_{n \in \{0, 1, \ldots, N^2\}}$
- $Y^N_n \in P_N(H)$ for all $n \in \{0, 1, \ldots, N^2\}$ and $P_N(H)$ is $N$-dimensional

$N^2 \cdot \frac{N}{\dim(P_N(H))} = N^3$ independent standard normal random variables needed to simulate $Y^N_{N^2} \approx X_T$
Theorem (J & Röckner, 2010)

There exist $C_r > 0$, $r \in (0, 2)$, such that

$$
\left( \mathbb{E} \left[ \int_0^1 |X_T(x) - Y_{N^2}^N(x)|^2 \, dx \right] \right)^{1/2} \leq C_r \cdot N^{(r-2)}
$$

holds for all $N \in \mathbb{N}$ and all arbitrarily small $r \in (0, 2)$.

- $Y_{N^2}^N$ converges to $X_T$ with order 2—
- $N^3$ random variables are needed to simulate $Y_{N^2}^N$

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Numerical example

Consider the SPDE

$$dX_t(x) = \left[ \frac{1}{20} \frac{\partial^2}{\partial x^2} X_t(x) + 1 - X_t(x) \right] dt + \frac{X_t(x)}{1 + X_t(x)^2} dW_t(x)$$

with $X_t(0) = X_t(1) = 0$ and $X_0(x) = 0$ for $x \in (0, 1)$ and $t \in [0, T]$ with $T = 1$.

We plot

$$\left( \mathbb{E} \left[ \int_0^1 \left| X_T(x) - Z_{N^4}^N(x) \right|^2 dx \right] \right)^{\frac{1}{2}}$$

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How is this result related to Müllер-Gronbach and Ritter’s complexity bound?
How is this result related to **Müller-Gronbach and Ritter’s complexity bound**?
Let $d = 2$ and consider the SPDE

$$
\frac{dX_t(x_1, x_2)}{dt} = \left[ \frac{1}{50} \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right) X_t(x_1, x_2) \right] dt + X_t(x_1, x_2) \, dW_t(x_1, x_2)
$$

with $X_t|_{\partial(0,1)^2} \equiv 0$ and $X_0(x_1, x_2) = 2 \sin(\pi x_1) \sin(\pi x_2)$ for $x_1, x_2 \in (0, 1)$ and $t \in [0, 1]$ with the covariance operator $Q : H \to H$ given by

$$
(Qv)(x_1, x_2) = \sum_{j_1, j_2 = 1}^{\infty} \frac{2 \sin(j_1 \pi x_1) \sin(j_2 \pi x_2)}{(j_1 + j_2)^4} \int_0^1 \int_0^1 \sin(j_1 \pi y_1) \sin(j_2 \pi y_2) \, v(y_1, y_2) \, dy_1 \, dy_2
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A. Jentzen
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A two-dimensional stochastic heat equation

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1. Taylor expansions for SODEs
2. Taylor expansions for SPDEs
3. A new numerical method for SPDEs with non-additive noise
4. A new numerical method for SPDEs with additive noise
Reconsider the SPDE and the Exponential Euler approximation

Reconsider the SPDE

\[ dX_t = \left[ AX_t + F(X_t) \right] dt + B(X_t) dW_t, \quad X_0 = \xi \]

for \( t \in [0, T] \). SPDE in the mild form

\[ X_t = e^{At}X_0 + \int_0^t e^{A(t-s)}F(X_s) \, ds + \int_0^t e^{A(t-s)}B(X_s) \, dW_s \]

for all \( t \in [0, T] \). Exponential Euler approximation:

\[ X_t \approx e^{At}X_0 + A^{-1}(e^{At} - I)F(X_0) + \int_0^t e^{A(t-s)}B(X_0) \, dW_s \]
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A stochastic Ginzburg-Landau PDE with additive space-time white noise

Consider the SPDE

\[ dX_t(x) = \left[ \Delta X_t(x) + X_t(x) - X_t(x)^3 \right] dt + dW_t(x) \]

with \( X_t(0) = X_t(1) = 0 \) and \( X_0 = 0 \) for \( x \in (0, 1) \) and \( t \in [0, T] \) on \( H = L^2((0, 1), \mathbb{R}) \) with \( T = 1 \) and where \( (W_t)_{t \in [0, T]} \) is a cylindrical \( l \)-Wiener process on \( H \) here.

**Goal:** Compute

\[ X_T(\omega, x), \quad x \in [0, 1], \]

with the precision of two decimals, i.e. with the precision \( \varepsilon = \frac{1}{100} \), for one random \( \omega \in \Omega \).
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with the precision of two decimals, i.e. with the precision \( \varepsilon = \frac{1}{100} \), for one random \( \omega \in \Omega \).
Consider the SPDE

\[ dX_t(x) = \left[ \Delta X_t(x) + X_t(x) - X_t(x)^3 \right] dt + dW_t(x) \]

with \( X_t(0) = X_t(1) = 0 \) and \( X_0 = 0 \) for \( x \in (0, 1) \) and \( t \in [0, T] \) on \( H = L^2((0, 1), \mathbb{R}) \) with \( T = 1 \) and where \( (W_t)_{t \in [0, T]} \) is a cylindrical \( \mathcal{L} \)-Wiener process on \( H \) here.

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Linear implicit Euler scheme and spectral Galerkin approximations

\[ Z_N^n : \Omega \rightarrow P_N(H), \ n \in \{0, 1, \ldots, N^2\}, \ N \in \mathbb{N}, \text{ given by } Z_0^N = 0 \text{ and } \]

\[ Z_N^{n+1} = \left( I - \frac{T}{N^2} A \right)^{-1} \left( Z_N^n + \frac{T}{N^2} \cdot (P_N F)(Z_N^n) + \int_{nT/N^2}^{(n+1)T/N^2} P_N \, dW_s \right) \]

\( \mathbb{P} \)-a.s. for all \( n \in \{0, 1, \ldots, N^2 - 1\} \) and all \( N \in \mathbb{N} \).

- \( N^2 \) time steps are used in \( (Z_N^n)_{n \in \{0,1,\ldots,N^2\}} \)
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Linear implicit Euler scheme and spectral Galerkin approximations

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\[
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\]

\[ \mathbb{P} \text{-a.s. for all } n \in \{0, 1, \ldots, N^2 - 1\} \text{ and all } N \in \mathbb{N}. \]

\begin{itemize}
  \item \(N^2\) time steps are used in \((Z_n^N)_{n \in \{0, 1, \ldots, N^2\}}\)
  \item \(Z_n^N \in P_N(H)\) for all \(n \in \{0, 1, \ldots, N^2\}\) and \(P_N(H)\) is \(N\)-dimensional
  \item \[
  \underbrace{N^2}_{\text{time steps}} \cdot \underbrace{N}_{\text{dim}(P_N(H))} = N^3
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\( N^2 \) independent standard normal random variables needed to simulate \( Z_{N^2}^N \approx X_T \).
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\[ N^2 \cdot \left\lceil \frac{N}{\dim(P_N(H))} \right\rceil = N^3 \] independent standard normal random variables needed to simulate \( Z^N_{N^2} \approx X_T \)
\( Y^N_n : \Omega \rightarrow P_{N^2}(H), \ n \in \{0, 1, \ldots, N\}, \ N \in \mathbb{N}, \) given by \( Y^N_0 = 0 \) and

\[
Y^N_{n+1} = e^{A\frac{T}{N}} Y^N_n + \left( e^{A\frac{T}{N}} - I \right) \cdot (P_{N^2} F)(Y^N_n) + \int_{\frac{nT}{N}}^{\frac{(n+1)T}{N}} P_{N^2} e^{A\left(\frac{(n+1)T}{N} - s\right)} dW_s
\]

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N \begin{array}{c} \text{time steps} \\ \hline \text{dim}(P_{N^2}(H)) \end{array} = N^3 \quad \text{independent standard normal random variables needed to simulate } Y^N_n \approx X_T
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Exponential Euler scheme

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\]

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$\mathbb{P}$-a.s. for all $n \in \{0, 1, \ldots, N - 1\}$ and all $N \in \mathbb{N}$.

- $N$ time steps are used in $(Y^N_n)_{n \in \{0, 1, \ldots, N\}}$
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\( \approx \) independent standard normal random variables needed to simulate \( Y_N^N \approx X_T \)
Exponential Euler scheme

\[ Y_N^n : \Omega \rightarrow P_{N^2}(H), \ n \in \{0, 1, \ldots, N\}, \ N \in \mathbb{N}, \ \text{given by} \ Y_0^N = 0 \ \text{and} \]

\[ Y_{n+1}^N = e^{A_T N} Y_n^N + \left( e^{A_T N^T} - I \right) \cdot (P_{N^2} F)(Y_n^N) + \int_{nT/N}^{(n+1)T/N} P_{N^2} e^{A((n+1)T/N - s)} dW_s \]

\( \mathbb{P} \)-a.s. for all \( n \in \{0, 1, \ldots, N - 1\} \) and all \( N \in \mathbb{N} \).

- \( N \) time steps are used in \( (Y_N^n)_{n \in \{0,1,\ldots,N\}} \)
- \( Y_N^n \in P_{N^2}(H) \) for all \( n \in \{0, 1, \ldots, N\} \) and \( P_{N^2}(H) \) is \( N^2 \)-dimensional

\[ \text{\( \sqrt{N} \) time steps} \cdot \sqrt{\text{dim}(P_{N^2}(H))} = N^3 \ \text{independent standard normal random variables needed to simulate} \ Y_N^N \approx X_T \]
Numerical results

We plot

$$\sup_{x \in [0,1]} \left| X_T(\omega, x) - Z^N_{N^2}(\omega, x) \right|$$

and

$$\sup_{x \in [0,1]} \left| X_T(\omega, x) - Y^N_N(\omega, x) \right|$$

for different $N \in \mathbb{N}$ and one random $\omega \in \Omega$. 
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for different \( N \in \mathbb{N} \) and one random \( \omega \in \Omega \).
Linear implicit Euler scheme: $Z_{N}^{N_{2}}$ with $N = 8192$ ($N^{3} = 8193^{3} \approx 0.5 \cdot 10^{12}$ random variables) achieves the desired precision $\varepsilon = \frac{1}{100}$.
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MATLAB code for $Z_{N^2}^N$ with $N = 8192$

```matlab
N = 8192; M = 67108864; A = -pi^2*(1:N).^2; Y = zeros(1,N);
for m=1:M
    y = dst(Y) * sqrt(2);
    Y = (Y + idst(y-y.^3)/sqrt(2)/M + randn(1,N)/sqrt(M))./(1 - A/M);
end
plot((0:N+1)/(N+1), [0, dst(Y)*sqrt(2), 0]);
```

CPU time on an INTEL PENTIUM D (3.0 GHz):

Linear implicit Euler scheme: \( \approx 6 \) days and 22 hours
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Exponential Euler scheme: $Y_N^{N}$ with $N = 64$ ($N^3 = 64^3 \approx 2.6 \cdot 10^5$ random variables) achieves the desired precision $\varepsilon = \frac{1}{100}$. 
A new numerical method for SPDEs with non-additive noise

A new numerical method for SPDEs with additive noise

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Taylor expansions for SODEs
Taylor expansions for SPDEs

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Precise number of used random variables

Approximation error

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    y = dst(Y) * sqrt(2);
    Y = (Y + idst(y-y.^3)/sqrt(2)/M + randn(1,N)/sqrt(M))./ (1 - A/M);
end
plot((0:N+1)/(N+1), [0, dst(Y)*sqrt(2), 0]);
```

CPU time on an INTEL PENTIUM D (3.0 GHz):

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MATLAB code for $Y_N^N$ with $N = 64$

```matlab
N = 8192; M = 67108864; A = −pi^2*(1:N).^2; Y = zeros(1,N);
Q = sqrt((exp(2*A/M)−1)/2./A);
for m=1:M
    y = dst(Y) * sqrt(2);
    Y = (Y + idst(y−y.^3)/sqrt(2)/M + randn(1,N)/sqrt(M))./(1 − A/M);
end
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Q = sqrt((exp(2*A/M)-1)/2./A);
for m=1:M
    y = dst(Y) * sqrt(2);
    Y = exp(A/M).*Y+(exp(A/M)-1)./A.*idst(y-y.^3)/sqrt(2)+Q.*randn(1,N);
end
plot((0:N+1)/(N+1), [0,dst(Y)*sqrt(2),0]);
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CPU time on an INTEL PENTIUM D (3.0 GHz):

Linear implicit Euler scheme: $\approx 6$ days and 22 hours
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1. $N = 4096$; $M = 67108864$; $A = -\pi^2*{(1:N)}^2$; $Y = \text{zeros}(1,N)$;
2. $Q = \sqrt{(\exp(2*A/M) - 1)/2./A)}$;
3. for $m = 1:M$
   4. $y = \text{dst}(Y) * \sqrt{2}$;
   5. $Y = \exp(A/M) * Y + (\exp(A/M) - 1)/A * \text{idst}(y - y.^3)/\sqrt{2} + Q * \text{randn}(1,N)$;
4. end
5. plot((0:N+1)/(N+1), [0, dst(Y) * sqrt(2), 0]);

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```matlab
N = 4096; M = 64; A = -pi^2*(1:N).^2; Y = zeros(1,N);
Q = sqrt((exp(2*A/M) - 1)/2./A);
for m=1:M
    y = dst(Y) * sqrt(2);
    Y = exp(A/M).*Y + (exp(A/M) - 1)./A.*idst(y-y.^3)/sqrt(2) + Q.*randn(1,N);
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```

CPU time on an INTEL PENTIUM D (3.0 GHz):

Linear implicit Euler scheme: $\approx$ 6 days and 22 hours

Exponential Euler scheme: $\approx$ 0.48 seconds
References: