Taylor Expansions and Numerical Approximations for Stochastic Partial Differential Equations

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Joint works with P. E. Kloeden and M. Röckner

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Content

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Consider the SODE:

$$dX_t = f(X_t)\,dt + g(X_t)\,dW_t,$$

which is understood as

$$X_t = X_0 + \int_0^t f(X_s)\,ds + \int_0^t g(X_s)\,dW_s$$

$\mathbb{P}$-a.s. for all $t \in [0, T]$. Applying Itô's formula to the integrands above yields

$$X_t \approx X_0 + f(X_0) \cdot t + g(X_0) \cdot \int_0^t dW_s + g'(X_0)\,g(X_0) \cdot \int_0^t \int_0^s dW_u\,dW_s$$

$$= X_0 + f(X_0) \cdot t + g(X_0) \cdot W_t + \frac{1}{2} \cdot g'(X_0)\,g(X_0) \cdot \left((W_t)^2 - t\right) \qquad \mathbb{P}\text{-a.s.}$$

Milstein's approximation (1974).

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The corresponding numerical scheme, the so-called **Milstein scheme** (or also Taylor scheme of (strong) order 1.0), is then given by $Y_0^N = X_0$ and

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for every $n \in \{0, 1, \ldots, N-1\}$, $N \in \mathbb{N}$ and is of (strong) order 1.

It is very easy to simulate and impressively efficient for one-dimensional SODEs in comparison to e.g. Euler's method which is of (strong) order $\frac{1}{2}$.

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The general theory of Taylor expansions for SODEs and the numerical schemes based on them can be found in the monographs

- Kloeden & Platen (1992)
- Milstein (1995)

(Stochastic Taylor expansions).

Essential constituents:

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Method for deriving Taylor expansions:

- Iterated application of the stochastic fundamental theorem of calculus, i.e. Itô's formula.

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Itô's formula for SPDEs? In general not a semi-martingale

Let $(H, \|\cdot\|)$ be a reasonable state space of a SPDE.

- Krylov & Rozovskii (1979): Itô formula for $F : H \to \mathbb{R}$, $F(v) = \|v\|^2$, $v \in H$, with powerful consequences to the variational approach, see also Gyöngy & Krylov (1981) and Prévot & Röckner (2007)
- Gradinaru, Nourdin & Tindel (2005): Itô formula for $F : H \to \mathbb{R}$ smooth
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For the Taylor expansion approach for SODEs, one would need an Itô formula for $F : H \to H$ smooth.

If the solution of the SPDE is spatially smooth and the solution is a semi-martingale, then Taylor expansions & Taylor schemes for SPDEs in

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SPDE Setting

Fix $T > 0$, a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with normal filtration $(\mathcal{F}_t)_{t\in[0,T]}$ and two $\mathbb{R}$-Hilbert spaces H and U.

(A1) Let $A : D(A) \subset H \to H$ be a bijective linear operator with negative compact inverse.

(A2) Let $W : [0, T] \times \Omega \to U$ be a standard Q-Wiener process with covariance operator $Q : U \to U$. Let $U_0 := Q^{\frac{1}{2}}(U)$.

(A3) Let $F : D((-A)^{\beta}) \to H$ and $B : D((-A)^{\beta}) \to HS(U_0, H)$ with $\beta \in [0, 1)$ be twice continuously Fréchet differentiable with appropriate globally bounded derivatives.

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Consider the SPDE

$$dX_t = \Big[AX_t + F(X_t)\Big]\,dt + B(X_t)\,dW_t, \qquad X_0 = \xi$$

for $t \in [0, T]$.

SPDE in the mild form

$$X_t = e^{At}\xi + \int_0^t e^{A(t-s)}F(X_s)\,ds + \int_0^t e^{A(t-s)}B(X_s)\,dW_s$$

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Example

Let $d \in \mathbb{N}$ and consider the SPDE

$$dX_t(x) = \Big[\kappa \Delta X_t(x) + f(x, X_t(x))\Big]\, dt + b(x, X_t(x))\, dW_t(x)$$

with $X_t|_{\partial(0,1)^d} \equiv 0$ and $X_0(x) = \xi(x)$ for $x \in (0,1)^d$ and $t \in [0, T]$. Here

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$\mathbb{P}$-a.s. for all $t \in [0, T]$. Subtracting X_0 gives

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$$\Delta X_t = \left(e^{At} - I\right) X_0 + \int_0^t e^{A(t-s)} F(X_s)\, ds + \int_0^t e^{A(t-s)} B(X_s)\, dW_s$$

$\mathbb{P}$-a.s. for all $t \in [0, T]$. Omitting first and second summand yields the first simple Taylor approximation for SPDEs

$$\Delta X_t \approx \int_0^t e^{A(t-s)} B(X_0)\, dW_s, \qquad \text{i.e.}$$

$$X_t \approx \mathbf{X_0} + \int_{\mathbf{0}}^{\mathbf{t}} \mathbf{e^{A(t-s)} B(X_0)\, dW_s}$$

Omitting second summand gives

$$\Delta X_t \approx \left(e^{At} - I\right) X_0 + \int_0^t e^{A(t-s)} B(X_0)\, dW_s, \qquad \text{i.e.}$$

$$X_t \approx \mathbf{e^{At} X_0} + \int_{\mathbf{0}}^{\mathbf{t}} \mathbf{e^{A(t-s)} B(X_0)\, dW_s}$$

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$\mathbb{P}$-a.s. for all $t \in [0, T]$. Classical Taylor approximations $F(X_s) \approx F(X_0)$ and $B(X_s) \approx B(X_0)$ yield

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$$X_t \approx e^{At} X_0 + A^{-1}\left(e^{At} - I\right) F(X_0) + \int_0^t e^{A(t-s)} B(X_0)\, dW_s$$

Exponential Euler approximation (see the final section for more details).

Consider

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Using $\Delta X_s \approx \int_0^s e^{A(s-u)} B(X_0)\, dW_u$ (first simple Taylor approximation) recursively yields

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$$X_t \approx \mathbf{e}^{\mathbf{A}\mathbf{t}} \mathbf{X}_0 + \int_0^t \mathbf{e}^{\mathbf{A}(\mathbf{t}-\mathbf{s})} \mathbf{F}(\mathbf{X}_0)\, \mathbf{ds} + \int_0^t \mathbf{e}^{\mathbf{A}(\mathbf{t}-\mathbf{s})} \mathbf{B}(\mathbf{X}_0)\, \mathbf{dW_s} + \int_0^t \mathbf{e}^{\mathbf{A}(\mathbf{t}-\mathbf{s})} \mathbf{B}'(\mathbf{X}_0) \left(\int_0^s \mathbf{e}^{\mathbf{A}(\mathbf{s}-\mathbf{u})} \mathbf{B}(\mathbf{X}_0)\, \mathbf{dW_u} \right) \mathbf{dW_s}$$

Infinite dimensional analog of Milstein's approximation (see the next section of this talk)

Using $\Delta X_s \approx \int_0^s e^{A(s-u)} B(X_0)\, dW_u$ (first simple Taylor approximation) recursively yields

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$$X_t \approx \mathbf{e^{At} X_0} + \int_{\mathbf{0}}^{\mathbf{t}} \mathbf{e^{A(t-s)} F(X_0)\, ds} + \int_{\mathbf{0}}^{\mathbf{t}} \mathbf{e^{A(t-s)} B(X_0)\, dW_s} + \int_{\mathbf{0}}^{\mathbf{t}} \mathbf{e^{A(t-s)} B'(X_0)} \left(\int_{\mathbf{0}}^{\mathbf{s}} \mathbf{e^{A(s-u)} B(X_0)\, dW_u} \right) \mathbf{dW_s}$$

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Infinite dimensional analog of Milstein's approximation (see the next section of this talk)

Iterating this idea yields more Taylor approximations such as

$$
\begin{aligned}
X_t \approx e^{At} X_0 &+ \int_0^t e^{A(t-s)} F(X_0)\, ds + \int_0^t e^{A(t-s)} B(X_0)\, dW_s \\
&+ \int_0^t e^{A(t-s)} B'(X_0) \left(e^{As} - I \right) X_0 \, dW_s \\
&+ \int_0^t e^{A(t-s)} B'(X_0) \left(\int_0^s e^{A(s-u)} B(X_0)\, dW_u \right) dW_s \\
&+ \frac{1}{2} \int_0^t e^{A(t-s)} B''(X_0) \left(\int_0^s e^{A(s-u)} B(X_0)\, dW_u , \int_0^s e^{A(s-u)} B(X_0)\, dW_u \right) dW_s
\end{aligned}
$$

Some remarks:

- References:
  - "Taylor expansions of solutions of stochastic partial differential equations with additive noise" (J & Kloeden; Ann. Probab. 2010)
  - "Taylor expansions of solutions of stochastic partial differential equations" (J; DCDS B 2010)
- Systematic theory with appropriate integral operators
- Precisely description of the remainder terms & arbitrarily high orders
- Essential constituents: $\int_0^t e^{A(t-s)} B'(X_0) \left(\int_0^s e^{A(s-u)} B(X_0)\, dW_u \right) dW_s$
- Method: classical Taylor expansions in Banach spaces & recursion technique
- These Taylor Expansions for SPDEs generalize the Taylor Expansions for ODEs and SODEs.

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Content

Reconsider the SPDE and the infinite dimensional analog of Milstein's approximation

Reconsider the SPDE

$$dX_t = \Big[AX_t + F(X_t)\Big]dt + B(X_t)\,dW_t, \qquad X_0 = \xi$$

for $t \in [0, T]$. SPDE in the mild form

$$X_t = e^{At}X_0 + \int_0^t e^{A(t-s)}F(X_s)\,ds + \int_0^t e^{A(t-s)}B(X_s)\,dW_s$$

$\mathbb{P}$-a.s. for all $t \in [0, T]$. Infinite dimensional analog of Milstein's approximation:

$$X_t \approx \mathbf{e^{At}X_0} + \int_0^{\mathbf{t}} \mathbf{e^{A(t-s)}F(X_0)\,ds} + \int_0^{\mathbf{t}} \mathbf{e^{A(t-s)}B(X_0)\,dW_s}$$
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Reconsider the SPDE and the infinite dimensional analog of Milstein's approximation

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for $t \in [0, T]$. SPDE in the mild form

$$X_t = e^{At} X_0 + \int_0^t e^{A(t-s)} F(X_s)\, ds + \int_0^t e^{A(t-s)} B(X_s)\, dW_s$$

$\mathbb{P}$-a.s. for all $t \in [0, T]$. Infinite dimensional analog of Milstein's approximation:

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Reconsider the SPDE and the infinite dimensional analog of Milstein's approximation

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A one-dimensional example SPDE

Let $d = 1$ and consider the SPDE

$$dX_t(x) = \left[\kappa \frac{\partial^2}{\partial x^2} X_t(x) + f(x, X_t(x))\right] dt + b(x, X_t(x))\, dW_t(x)$$

with $X_t(0) = X_t(1) = 0$ and $X_0(x) = \xi(x)$ for $x \in (0,1)$ and $t \in [0,T]$ with the covariance operator $Q : H \to H$ given by

$$(Qv)(x) = \sum_{j=1}^{\infty} \frac{2}{j^3} \cos(j\pi x) \int_0^1 \cos(j\pi y)\, v(y)\, dy$$

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Goal: Solve the strong approximation problem, i.e. compute $Y : \Omega \to H$ such that

$$\left(\mathbb{E}\left[\int_0^1 |X_T(x) - Y(x)|^2 \, dx \right] \right)^{\frac{1}{2}} < \varepsilon$$

holds for a given precision $\varepsilon > 0$ with the least possible computational effort.

Spectral Galerkin approximations: $P_N : H \to H$, $N \in \mathbb{N}$, given by

$$(P_N(v))(x) = \sum_{n=1}^{N} 2 \sin(n\pi x) \int_0^1 \sin(n\pi y) \, v(y) \, dy$$

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Linear implicit Euler scheme and spectral Galerkin approximations

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for all $n \in \{0, 1, \ldots, N^4 - 1\}$, $N \in \mathbb{N}$.

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Theorem (e.g. Hausenblas, 2003)

There exist $C_r > 0$, $r \in (0,2)$, such that

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A new algorithm for SPDEs with non-additive noise

Reconsider the infinite dimensional analog of Milstein's approximation

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Theorem (J & Röckner, 2010)

There exist $C_r > 0$, $r \in (0,2)$, such that

$$\left(\mathbb{E}\left[\int_0^1 \left| X_T(x) - Y^N_{N^2}(x) \right|^2 dx \right] \right)^{\frac{1}{2}} \leq C_r \cdot N^{(r-2)}$$

holds for all $N \in \mathbb{N}$ and all arbitrarily small $r \in (0,2)$.

- $Y^N_{N^2}$ converges to X_T with order $2-$
- N^3 random variables are needed to simulate $Y^N_{N^2}$
- Conclusion: about $O(\varepsilon^{-\frac{3}{2}})$ random variables are needed to achieve the desired precision $\varepsilon > 0$

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Numerical example

Consider the SPDE

$$dX_t(x) = \left[\frac{1}{20} \frac{\partial^2}{\partial x^2} X_t(x) + 1 - X_t(x) \right] dt + \frac{X_t(x)}{1 + X_t(x)^2} \, dW_t(x)$$

with $X_t(0) = X_t(1) = 0$ and $X_0(x) = 0$ for $x \in (0, 1)$ and $t \in [0, T]$ with $T = 1$.

We plot

$$\left(\mathbb{E} \left[\int_0^1 \left| X_T(x) - Z^N_{N^4}(x) \right|^2 dx \right] \right)^{\frac{1}{2}}$$

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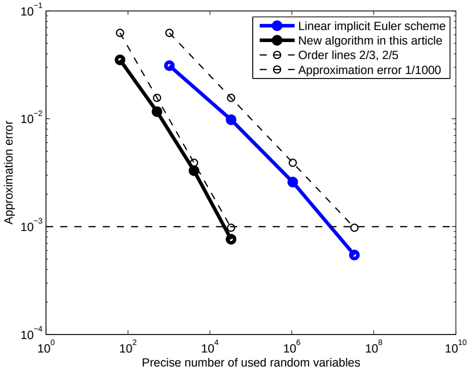

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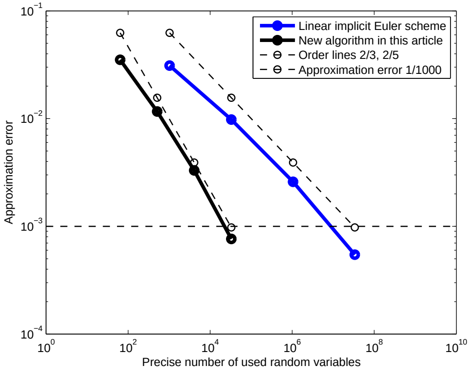

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A two-dimensional stochastic heat equation

Let $d = 2$ and consider the SPDE

$$dX_t(x_1, x_2) = \left[\frac{1}{50} \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right) X_t(x_1, x_2) \right] dt + X_t(x_1, x_2) \, dW_t(x_1, x_2)$$

with $X_t|_{\partial(0,1)^2} \equiv 0$ and $X_0(x_1, x_2) = 2 \sin(\pi x_1) \sin(\pi x_2)$ for $x_1, x_2 \in (0, 1)$ and $t \in [0, 1]$ with the covariance operator $Q : H \to H$ given by

$$(Qv)(x_1, x_2) = \sum_{j_1, j_2 = 1}^{\infty} \frac{2 \sin(j_1 \pi x_1) \sin(j_2 \pi x_2)}{(j_1 + j_2)^4} \int_0^1 \int_0^1 \sin(j_1 \pi y_1) \sin(j_2 \pi y_2) \, v(y_1, y_2) \, dy_1 \, dy_2$$

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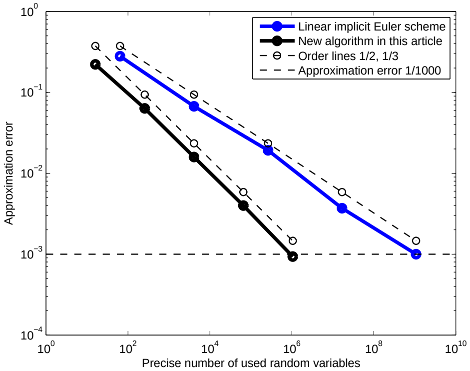

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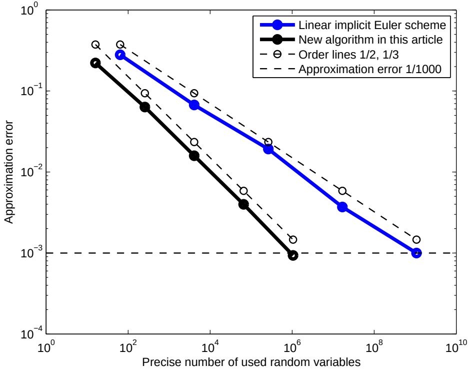

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Content

Reconsider the SPDE and the Exponential Euler approximation

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$$dX_t = \left[AX_t + F(X_t)\right]dt + B(X_t)\,dW_t, \qquad X_0 = \xi$$

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for all $t \in [0, T]$. Exponential Euler approximation:

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A stochastic Ginzburg-Landau PDE with additive space-time white noise

Consider the SPDE

$$dX_t(x) = \left[\Delta X_t(x) + X_t(x) - X_t(x)^3\right] dt + dW_t(x)$$

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Goal: Compute

$$X_T(\omega, x), \quad x \in [0,1],$$

with the precision of two decimals, i.e. with the precision $\varepsilon = \frac{1}{100}$, for one random $\omega \in \Omega$.

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Linear implicit Euler scheme and spectral Galerkin approximations

$Z_n^N : \Omega \to P_N(H)$, $n \in \{0, 1, \ldots, N^2\}$, $N \in \mathbb{N}$, given by $Z_0^N = 0$ and

$$Z_{n+1}^N = \left(I - \frac{T}{N^2} A \right)^{-1} \left(Z_n^N + \frac{T}{N^2} \cdot (P_N F)(Z_n^N) + \int_{\frac{nT}{N^2}}^{\frac{(n+1)T}{N^2}} P_N \, dW_s \right)$$

$\mathbb{P}$-a.s. for all $n \in \{0, 1, \ldots, N^2 - 1\}$ and all $N \in \mathbb{N}$.

- N^2 time steps are used in $(Z_n^N)_{n \in \{0,1,\ldots,N^2\}}$
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Numerical results

We plot

$$\sup_{x \in [0,1]} \left| X_T(\omega, x) - Z^N_{N^2}(\omega, x) \right|$$

and

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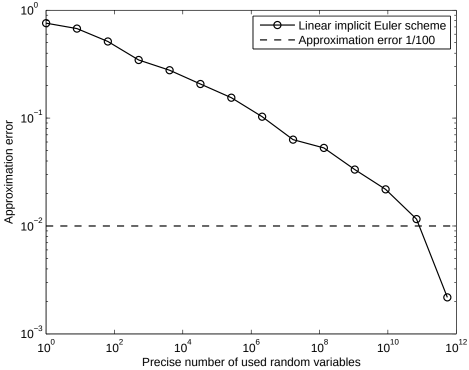
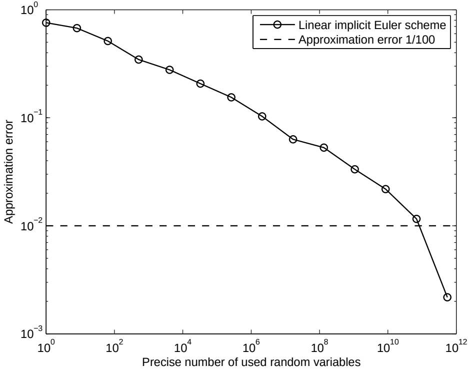


Linear implicit Euler scheme: $Z^N_{N^2}$ with $N = 8192$ ($N^3 = 8193^3 \approx 0.5 \cdot 10^{12}$ random variables) achieves the desired precision $\varepsilon = \frac{1}{100}$



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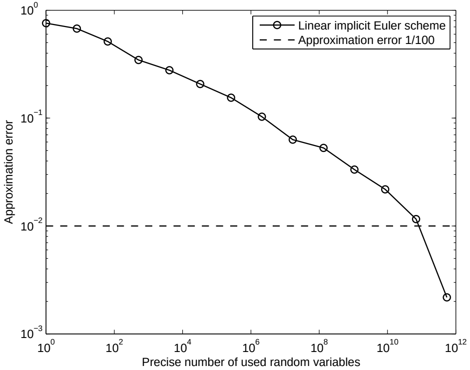

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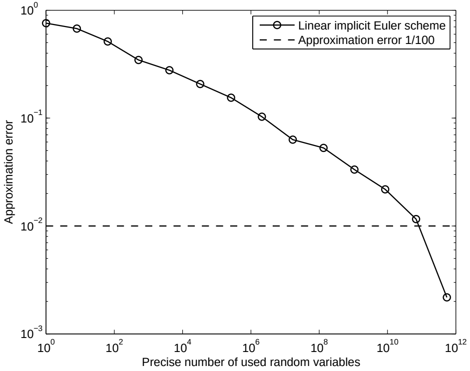

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MATLAB code for $Z^{N}_{N^2}$ with $N = 8192$

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for m=1:M
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   Y = (Y + idst(y-y.^3)/sqrt(2)/M + randn(1,N)/sqrt(M))./(1 - A/M);
end
plot( (0:N+1)/(N+1), [0,dst(Y)*sqrt(2),0] );
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CPU time on an INTEL PENTIUM D (3.0 GHz):

Linear implicit Euler scheme: $\approx$ **6 days and 22 hours**

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   Y = (Y + idst(y-y.^3)/sqrt(2)/M + randn(1,N)/sqrt(M))./(1 - A/M);
end
plot( (0:N+1)/(N+1), [0,dst(Y)*sqrt(2),0] );
```

CPU time on an INTEL PENTIUM D (3.0 GHz):

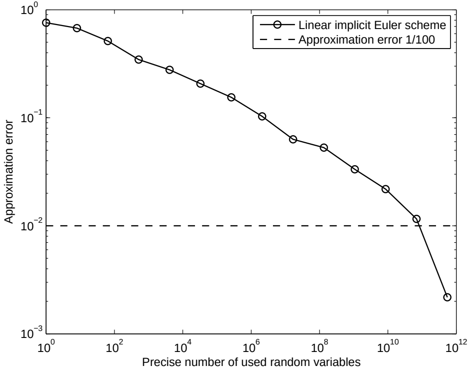
Linear implicit Euler scheme: $\approx$ **6 days and 22 hours**

MATLAB code for $Z_{N^2}^N$ with $N = 8192$

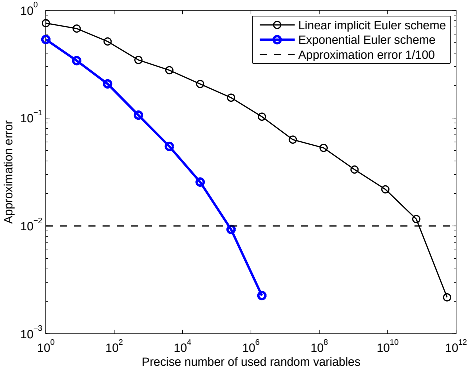
```
N = 8192; M = 67108864; A = -pi^2*(1:N).^2; Y = zeros(1,N);
for m=1:M
   y = dst(Y) * sqrt(2);
   Y = (Y + idst(y-y.^3)/sqrt(2)/M + randn(1,N)/sqrt(M))./(1 - A/M);
end
plot( (0:N+1)/(N+1), [0,dst(Y)*sqrt(2),0] );
```

CPU time on an INTEL PENTIUM D (3.0 GHz):

Linear implicit Euler scheme: $\approx$ **6 days and 22 hours**

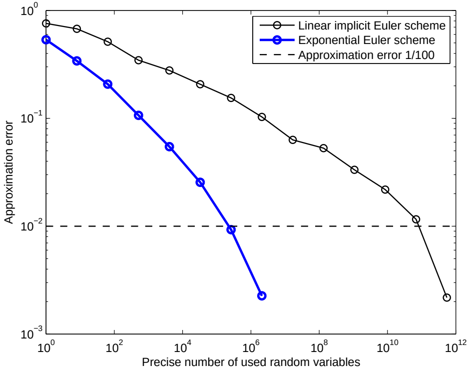


Linear implicit Euler scheme: $Z^N_{N^2}$ with $N = 8192$ ($N^3 = 8193^3 \approx 0.5 \cdot 10^{12}$ random variables) achieves the desired precision $\varepsilon = \frac{1}{100}$



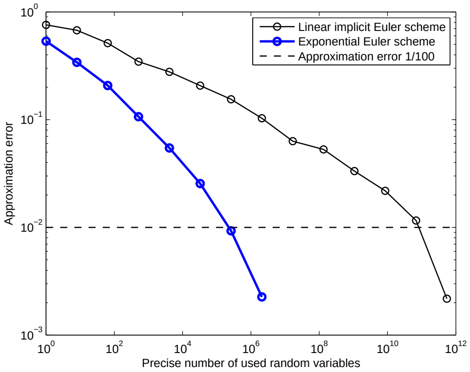
Linear implicit Euler scheme: $Z^N_{N^2}$ with $N = 8192$ ($N^3 = 8193^3 \approx 0.5 \cdot 10^{12}$ random variables) achieves the desired precision $\varepsilon = \frac{1}{100}$

Exponential Euler scheme: Y^N_N with $N = 64$ ($N^3 = 64^3 \approx 2.6 \cdot 10^5$ random variables) achieves the desired precision $\varepsilon = \frac{1}{100}$

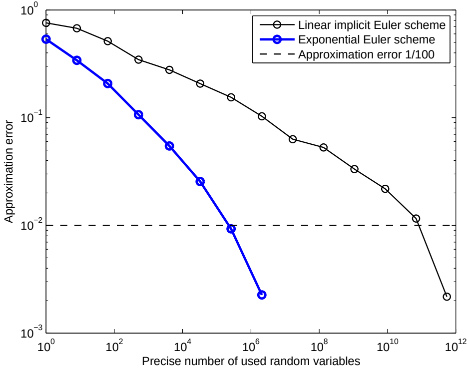


Linear implicit Euler scheme: $Z^N_{N^2}$ with $N = 8192$ ($N^3 = 8193^3 \approx 0.5 \cdot 10^{12}$ random variables) achieves the desired precision $\varepsilon = \frac{1}{100}$

Exponential Euler scheme: Y^N_N with $N = 64$ ($N^3 = 64^3 \approx 2.6 \cdot 10^5$ random variables) achieves the desired precision $\varepsilon = \frac{1}{100}$



Linear implicit Euler scheme: $Z^N_{N^2}$ with $N = 8192$ ($N^3 = 8193^3 \approx 0.5 \cdot 10^{12}$ random variables) achieves the desired precision $\varepsilon = \frac{1}{100}$
Exponential Euler scheme: Y^N_N with $N = 64$ ($N^3 = 64^3 \approx 2.6 \cdot 10^5$ random variables) achieves the desired precision $\varepsilon = \frac{1}{100}$



Linear implicit Euler scheme: $Z^N_{N^2}$ with $N = 8192$ ($N^3 = 8193^3 \approx 0.5 \cdot 10^{12}$ random variables) achieves the desired precision $\varepsilon = \frac{1}{100}$
Exponential Euler scheme: Y^N_N with $N = 64$ ($N^3 = 64^3 \approx 2.6 \cdot 10^5$ random variables) achieves the desired precision $\varepsilon = \frac{1}{100}$

MATLAB code for Y_N^N with $N = 64$

```
N = 8192; M = 67108864; A = -pi^2*(1:N).^2; Y = zeros(1,N);
 for m=1:M
   y = dst(Y) * sqrt(2);
   Y = (Y + idst(y-y.^3)/sqrt(2)/M + randn(1,N)/sqrt(M))./(1 - A/M);
end
 plot( (0:N+1)/(N+1), [0,dst(Y)*sqrt(2),0] );
```

CPU time on an INTEL PENTIUM D (3.0 GHz):

Linear implicit Euler scheme: $\approx$ **6 days and 22 hours**

MATLAB code for Y_N^N with $N = 64$

```
N = 8192; M = 67108864; A = -pi^2*(1:N).^2; Y = zeros(1,N);
Q = sqrt((exp(2*A/M) - 1)/2./A);
for m=1:M
   y = dst(Y) * sqrt(2);
   Y = (Y + idst(y-y.^3)/sqrt(2)/M + randn(1,N)/sqrt(M))./(1 - A/M);
end
plot( (0:N+1)/(N+1), [0,dst(Y)*sqrt(2),0] );
```

CPU time on an INTEL PENTIUM D (3.0 GHz):

Linear implicit Euler scheme: $\approx$ **6 days and 22 hours**

MATLAB code for Y_N^N with $N = 64$

```
N = 8192; M = 67108864; A = -pi^2*(1:N).^2; Y = zeros(1,N);
Q = sqrt((exp(2*A/M) - 1)/2./A);
for m=1:M
   y = dst(Y) * sqrt(2);
   Y = exp(A/M).*Y+(exp(A/M) - 1)./A.*idst(y-y.^3)/sqrt(2)+Q.*randn(1,N);
end
plot( (0:N+1)/(N+1), [0,dst(Y)*sqrt(2),0] );
```

CPU time on an INTEL PENTIUM D (3.0 GHz):

Linear implicit Euler scheme: $\approx$ **6 days and 22 hours**

MATLAB code for Y_N^N with $N = 64$

```
N = 4096; M = 67108864; A = -pi^2*(1:N).^2; Y = zeros(1,N);
Q = sqrt((exp(2*A/M) - 1)/2./A);
for m=1:M
   y = dst(Y) * sqrt(2);
   Y = exp(A/M).*Y+(exp(A/M) - 1)./A.*idst(y-y.^3)/sqrt(2)+Q.*randn(1,N);
end
plot( (0:N+1)/(N+1), [0,dst(Y)*sqrt(2),0] );
```

CPU time on an INTEL PENTIUM D (3.0 GHz):

Linear implicit Euler scheme: $\approx$ **6 days and 22 hours**

MATLAB code for Y_N^N with $N = 64$

```
N = 4096; M = 64; A = -pi^2*(1:N).^2; Y = zeros(1,N);
Q = sqrt((exp(2*A/M) - 1)/2./A);
for m=1:M
   y = dst(Y) * sqrt(2);
   Y = exp(A/M).*Y+(exp(A/M) - 1)./A.*idst(y-y.^3)/sqrt(2)+Q.*randn(1,N);
end
plot( (0:N+1)/(N+1), [0,dst(Y)*sqrt(2),0] );
```

CPU time on an INTEL PENTIUM D (3.0 GHz):

Linear implicit Euler scheme: $\approx$ **6 days and 22 hours**

MATLAB code for Y_N^N with $N = 64$

```
N = 4096; M = 64; A = -pi^2*(1:N).^2; Y = zeros(1,N);
Q = sqrt((exp(2*A/M)-1)/2./A);
for m=1:M
   y = dst(Y) * sqrt(2);
   Y = exp(A/M).*Y+(exp(A/M)-1)./A.*idst(y-y.^3)/sqrt(2)+Q.*randn(1,N);
end
plot( (0:N+1)/(N+1), [0,dst(Y)*sqrt(2),0] );
```

CPU time on an INTEL PENTIUM D (3.0 GHz):

Linear implicit Euler scheme: $\approx$ **6 days and 22 hours**

Exponential Euler scheme: $\approx$ **0.48 seconds**

References:

- “Overcoming the order barrier in the numerical approximation of stochastic partial differential equations with additive space-time noise” (J & Kloeden; Proc. Roy. Soc. Lond. A 2009)
- “Efficient simulation of nonlinear parabolic SPDEs with additive noise” (J, Kloeden & Winkel; Ann. Appl. Probab. 2010)