On the global Lipschitz assumption in Computational Stochastics

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Overview

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Consider
- a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a normal filtration $(\mathcal{F}_t)_{t\in[0,T]}$ and $T > 0$
- a standard $(\mathcal{F}_t)_{t\in[0,T]}$-Brownian motion $W : [0,T] \times \Omega \to \mathbb{R}$
- continuous functions $\mu, \sigma : \mathbb{R} \to \mathbb{R}$ and
- a $\mathcal{F}_0/\mathcal{B}(\mathbb{R})$-measurable mapping $\xi : \Omega \to \mathbb{R}$ with $\mathbb{E}|\xi|^p < \infty \, \forall \, p \in [1,\infty)$.

Then let $X : [0,T] \times \Omega \to \mathbb{R}$ be an adapted stochastic process with continuous sample paths which fulfills

$$X_t = \xi + \int_0^t \mu(X_s)\, ds + \int_0^t \sigma(X_s)\, dW_s \qquad \mathbb{P}\text{-a.s.}$$

for all $t \in [0,T]$. Short form:

$$dX_t = \mu(X_t)\, dt + \sigma(X_t)\, dW_t, \qquad X_0 = \xi, \qquad t \in [0,T].$$

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Examples of SDEs I

Black-Scholes model with $\bar{\mu}, \bar{\sigma}, x_0 \in (0, \infty)$:

$$dX_t = \bar{\mu}\, X_t\, dt + \bar{\sigma}\, X_t\, dW_t, \qquad X_0 = x_0, \qquad t \in [0, T]$$

A SDE with a cubic drift and additive noise:

$$dX_t = -X_t^3\, dt + dW_t, \qquad X_0 = 0, \qquad t \in [0, 1]$$

A SDE with a cubic drift and multiplicative noise:

$$dX_t = -X_t^3\, dt + 6\, X_t \circ dW_t, \qquad X_0 = 1, \qquad t \in [0, 3]$$

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Examples of SDEs II

A stochastic Verhulst equation with $\eta, x_0 \in (0, \infty)$:

$$dX_t = X_t \left(\eta - X_t\right) dt + X_t \, dW_t, \quad X_0 = x_0, \quad t \in [0, T]$$

A Feller diffusion with logistic growth with $\eta, x_0 \in (0, \infty)$:

$$dX_t = X_t \left(\eta - X_t\right) dt + \sqrt{X_t} \, dW_t, \qquad X_0 = x_0, \qquad t \in [0, T]$$

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Weak approximation problem of the SDE (see, e.g., Kloeden & Platen (1992))

Suppose we want to compute

$$\mathbb{E}\Big[f(X_T)\Big]$$

for a given smooth function $f : \mathbb{R} \to \mathbb{R}$ whose derivatives grow at most polynomially.

For instance, $f(x) = x^2$ for all $x \in \mathbb{R}$ and we want to compute

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Approximation of $\mathbb{E}\big[f(X_T)\big]$

The **stochastic Euler scheme** $Y_k^N : \Omega \to \mathbb{R}$, $k \in \{0, 1, \ldots, N\}$, $N \in \mathbb{N}$, is given by $Y_0^N = \xi$ and

$$Y_{k+1}^N = Y_k^N + \frac{T}{N} \cdot \mu(Y_k^N) + \sigma(Y_k^N) \cdot \Big(W_{\frac{(k+1)T}{N}} - W_{\frac{kT}{N}}\Big)$$

for all $k \in \{0, 1, \ldots, N-1\}$ and all $N \in \mathbb{N}$.
Let $Y_k^{N,m} : \Omega \to \mathbb{R}$, $k \in \{0, 1, \ldots, N\}$, $N \in \mathbb{N}$, for $m \in \mathbb{N}$ be independent copies of the Euler approximations. The **Monte Carlo Euler approximation** with $N \in \mathbb{N}$ time steps and $M \in \mathbb{N}$ Monte Carlo runs is then given by

$$\frac{1}{M}\left(\sum_{m=1}^{M} f(Y_N^{N,m})\right) \approx \mathbb{E}\big[f(Y_N^N)\big] \approx \mathbb{E}\big[f(X_T)\big].$$

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The **stochastic Euler scheme** $Y_k^N \colon \Omega \to \mathbb{R}$, $k \in \{0, 1, \ldots, N\}$, $N \in \mathbb{N}$, is given by $Y_0^N = \xi$ and

$$Y_{k+1}^N = Y_k^N + \frac{T}{N} \cdot \mu\big(Y_k^N\big) + \sigma\big(Y_k^N\big) \cdot \Big(W_{\frac{(k+1)T}{N}} - W_{\frac{kT}{N}}\Big)$$

for all $k \in \{0, 1, \ldots, N-1\}$ and all $N \in \mathbb{N}$.
Let $Y_k^{N,m} \colon \Omega \to \mathbb{R}$, $k \in \{0, 1, \ldots, N\}$, $N \in \mathbb{N}$, for $m \in \mathbb{N}$ be independent copies of the Euler approximations. The **Monte Carlo Euler approximation** with $N \in \mathbb{N}$ time steps and $N^2 \in \mathbb{N}$ Monte Carlo runs is then given by

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Overview

The triangle inequality shows

$$\underbrace{\left| \mathbb{E}\Big[f\big(X_T\big) \Big] - \frac{1}{N^2} \sum_{m=1}^{N^2} f\big(Y_N^{N,m}\big) \right|}_{\text{error of the Monte Carlo Euler method}}$$

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holds for every smooth function $f : \mathbb{R} \to \mathbb{R}$ whose derivatives have at most polynomial growth (see, e.g., Kloeden & Platen (1992), Milstein (1995), Talay (1996), Higham (2001), Rössler (2003)).

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Numerically weak convergence

Theorem (see, e.g., Kloeden & Platen (1992))

Let $b, \sigma, f : \mathbb{R} \to \mathbb{R}$ be four times continuously differentiable with at most polynomially growing derivatives. Moreover, let $\mu, \sigma : \mathbb{R} \to \mathbb{R}$ be ***globally Lipschitz continuous****. Then there is a real number $C > 0$ such that*

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The stochastic Euler scheme converges in the numerically weak sense if the coefficients of the SDE are smooth and globally Lipschitz continuous.

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The Monte Carlo Euler method converges if the coefficients of the SDE are smooth and globally Lipschitz continuous.

Examples of SDEs I

The global Lipschitz assumption on the coefficients of the SDE is a serious shortcoming:

Black-Scholes model with $\bar{\mu}, \bar{\sigma}, x_0 \in (0, \infty)$:

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Overview

Open problem

Convergence of Euler's method

$$\lim_{N\to\infty} \mathbb{E}\left|X_T - Y_N^N\right| = 0, \quad \lim_{N\to\infty} \left|\mathbb{E}\left[(X_T)^2\right] - \mathbb{E}\left[\left(Y_N^N\right)^2\right]\right| = 0$$

for SDEs with superlinearly growing coefficients such as

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Higham, Mao and Stuart (2002) showed a conditional result: If Euler's method has bounded moments

$$\sup_{N\in\mathbb{N}} \mathbb{E}\left[\sup_{0\leq n\leq N} \left|Y_n^N\right|^{(2+\varepsilon)}\right] < \infty$$

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Divergence of Euler's method

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holds for:

A SDE with a cubic drift and additive noise:

$$dX_t = -X_t^3\,dt + dW_t, \qquad X_0 = 0, \qquad t \in [0,1]$$

A SDE with a cubic drift and multiplicative noise:

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Proof: Define "event of instability"

$$\Omega_N := \left\{ \omega \in \Omega \,\middle|\, \sup_{k \in \{1,2,\dots,N-1\}} \left| W_{\frac{k+1}{N}}(\omega) - W_{\frac{k}{N}}(\omega) \right| \leq 1, \right.$$

$$\left. \left| W_{\frac{1}{N}}(\omega) - W_0(\omega) \right| \geq 3N \right\}$$

for every $N \in \mathbb{N}$. Claim:

$$\left| \mathbf{Y}_{\mathbf{k}}^{\mathbf{N}}(\omega) \right| \geq (\mathbf{3N})^{\left(\mathbf{2}^{(\mathbf{k}-\mathbf{1})}\right)} \qquad \forall\, \mathbf{k} \in \{\mathbf{1}, \mathbf{2}, \dots, \mathbf{N}\} \tag{1}$$

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We fix $N \in \mathbb{N}$, $\omega \in \Omega_N$ and show (1) by induction on $k \in \{1, 2, \dots, N\}$.

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$$= \left| W_{\frac{1}{N}}(\omega) - W_0(\omega) \right| \geq 3N$$

Proof: Define "event of instability"

$$\Omega_N := \left\{ \omega \in \Omega \,\middle|\, \sup_{k \in \{1,2,\dots,N-1\}} \left| W_{\frac{k+1}{N}}(\omega) - W_{\frac{k}{N}}(\omega) \right| \leq 1, \right.$$
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for every $N \in \mathbb{N}$. Claim:

$$\left| \mathbf{Y}^{\mathbf{N}}_{\mathbf{k}}(\omega) \right| \geq (\mathbf{3N})^{\left(\mathbf{2}^{(\mathbf{k}-\mathbf{1})}\right)} \qquad \forall\, \mathbf{k} \in \{\mathbf{1}, \mathbf{2}, \dots, \mathbf{N}\} \tag{1}$$

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Induction hypothesis $\left|\mathbf{Y}_{\mathbf{k}}^{\mathbf{N}}(\omega)\right| \geq (\mathbf{3N})^{\left(\mathbf{2}^{(\mathbf{k}-\mathbf{1})}\right)}$ for one $k \in \{1, 2, \ldots, N\}$:

$$\begin{aligned}
\left|Y_{k+1}^N(\omega)\right| &= \left|Y_k^N(\omega) - \frac{1}{N}\left(Y_k^N(\omega)\right)^3 + \left(W_{\frac{k+1}{N}}(\omega) - W_{\frac{k}{N}}(\omega)\right)\right| \\
&\geq \left|\frac{1}{N}\left(Y_k^N(\omega)\right)^3\right| - \left|Y_k^N(\omega)\right| - \left|W_{\frac{k+1}{N}}(\omega) - W_{\frac{k}{N}}(\omega)\right| \\
&\geq \frac{1}{N}\left|Y_k^N(\omega)\right|^3 - \left|Y_k^N(\omega)\right| - 1 \\
&\geq \frac{1}{N}\left|Y_k^N(\omega)\right|^3 - 2\left|Y_k^N(\omega)\right|^2 \\
&= \left|Y_k^N(\omega)\right|^2\left(\frac{1}{N}\left|Y_k^N(\omega)\right| - 2\right) \\
&\geq \left|Y_k^N(\omega)\right|^2\left(\frac{1}{N}3N - 2\right) = \left|Y_k^N(\omega)\right|^2 \\
&\geq \left((3N)^{\left(2^{k-1}\right)}\right)^2 = (3N)^{\left(2^k\right)}
\end{aligned}$$

Induction hypothesis $\left|\mathbf{Y}^{\mathbf{N}}_{\mathbf{k}}(\omega)\right| \geq (\mathbf{3N})^{\left(\mathbf{2}^{(\mathbf{k}-\mathbf{1})}\right)}$ for one $k \in \{1, 2, \ldots, N\}$:

$$
\begin{aligned}
\left|Y^N_{k+1}(\omega)\right| &= \left|Y^N_k(\omega) - \frac{1}{N}\left(Y^N_k(\omega)\right)^3 + \left(W_{\frac{k+1}{N}}(\omega) - W_{\frac{k}{N}}(\omega)\right)\right| \\
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$$\begin{aligned}
\left|Y_{k+1}^{N}(\omega)\right| &= \left|Y_k^N(\omega) - \frac{1}{N}\left(Y_k^N(\omega)\right)^3 + \left(W_{\frac{k+1}{N}}(\omega) - W_{\frac{k}{N}}(\omega)\right)\right| \\
&\geq \left|\frac{1}{N}\left(Y_k^N(\omega)\right)^3\right| - \left|Y_k^N(\omega)\right| - \left|W_{\frac{k+1}{N}}(\omega) - W_{\frac{k}{N}}(\omega)\right| \\
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&= \left|Y_k^N(\omega)\right|^2\left(\frac{1}{N}\left|Y_k^N(\omega)\right| - 2\right) \\
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\left|Y^N_{k+1}(\omega)\right| &= \left|Y^N_k(\omega) - \frac{1}{N}\left(Y^N_k(\omega)\right)^3 + \left(W_{\frac{k+1}{N}}(\omega) - W_{\frac{k}{N}}(\omega)\right)\right| \\
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&\geq \left((3N)^{\left(2^{k-1}\right)}\right)^2 = (3N)^{\left(2^k\right)}
\end{aligned}$$

Induction hypothesis $|\mathbf{Y}^{\mathbf{N}}_{\mathbf{k}}(\omega)| \geq (\mathbf{3N})^{\left(\mathbf{2}^{(\mathbf{k}-\mathbf{1})}\right)}$ for one $k \in \{1, 2, \ldots, N\}$:

$$
\begin{aligned}
\left|Y^N_{k+1}(\omega)\right| &= \left| Y^N_k(\omega) - \frac{1}{N}\left(Y^N_k(\omega)\right)^3 + \left(W_{\frac{k+1}{N}}(\omega) - W_{\frac{k}{N}}(\omega)\right)\right| \\
&\geq \left|\frac{1}{N}\left(Y^N_k(\omega)\right)^3\right| - \left|Y^N_k(\omega)\right| - \left|W_{\frac{k+1}{N}}(\omega) - W_{\frac{k}{N}}(\omega)\right| \\
&\geq \frac{1}{N}\left|Y^N_k(\omega)\right|^3 - \left|Y^N_k(\omega)\right| - 1 \\
&\geq \frac{1}{N}\left|Y^N_k(\omega)\right|^3 - 2\left|Y^N_k(\omega)\right|^2 \\
&= \left|Y^N_k(\omega)\right|^2 \left(\frac{1}{N}\left|Y^N_k(\omega)\right| - 2\right) \\
&\geq \left|Y^N_k(\omega)\right|^2 \left(\frac{1}{N} 3N - 2\right) = \left|Y^N_k(\omega)\right|^2 \\
&\geq \left((3N)^{\left(2^{k-1}\right)}\right)^2 = (3N)^{\left(2^k\right)}
\end{aligned}
$$

In particular, we obtain

$$\left|Y_N^N(\omega)\right| \geq (3N)^{\left(2^{(N-1)}\right)} \tag{2}$$

for all $\omega \in \Omega_N$ and all $N \in \mathbb{N}$. Recall that

$$\Omega_N = \left\{ \omega \in \Omega \,\middle|\, \sup_{k \in \{1,\ldots,N-1\}} \left| W_{\frac{k+1}{N}}(\omega) - W_{\frac{k}{N}}(\omega) \right| \leq 1, \right.$$
$$\left. \left| W_{\frac{1}{N}}(\omega) - W_0(\omega) \right| \geq 3N \right\}$$

holds and therefore

$$\mathbb{P}\left[\Omega_N\right] \geq e^{-cN^2} \tag{3}$$

for all $N \in \mathbb{N}$ with $c \in (0, \infty)$ appropriate. Combining (2) and (3) shows

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Simulations of the first absolute moment of the solution of a SDE

Consider the SDE

$$dX_t = -10\,\mathrm{sgn}(X_t)\,|X_t|^{1.1}\,dt + 4\,dW_t\,, \qquad X_0 = 0, \qquad t \in [0, 10].$$

The first absolute moment of X_T with $T = 10$ satisfies

$$\mathbb{E}\Big[|X_{10}|\Big] \approx 0.7141\,.$$

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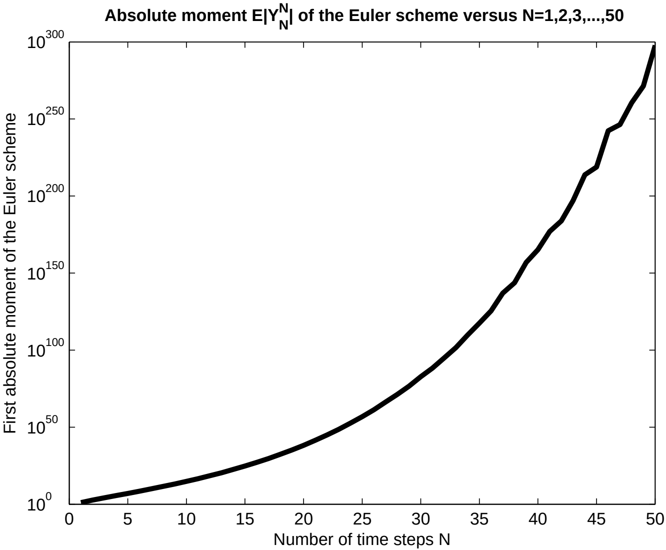
Absolute moment E|Y_N^N| of the Euler scheme versus N=1,2,3,...,50
First absolute moment of the Euler scheme
10^0
10^50
10^100
10^150
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10^250
10^300
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5
10
15
20
25
30
35
40
45
50
Number of time steps N

Simulations for a SDE with a cubic drift and multiplicative noise

Consider the SDE

$$dX_t = -X_t^3\,dt + 6\,X_t \circ dW_t\,, \qquad X_0 = 1, \qquad t \in [0,3].$$

The second moment of X_T with $T = 3$ satisfies

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Different simulation values of the Monte Carlo Euler method with 300 time steps and 10 000 Monte Carlo runs:

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Do we need **new numerical methods** which converge in the numerically weak sense?

Central observation: Numerically weak convergence fails to hold, i.e.

$$\lim_{N\to\infty} \mathbb{E}\Big[\big(Y_N^N\big)^2\Big] = \infty$$

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Theorem (Hutzenthaler & J (2009))

*Suppose that $\mu, \sigma, f \colon \mathbb{R} \to \mathbb{R}$ are four times continuously differentiable functions with at most polynomially growing derivatives. Moreover, let σ be globally Lipschitz continuous and let μ be **globally one-sided Lipschitz continuous**, i.e.*

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Conclusion

We should **not** split the error of the Monte Carlo Euler method

$$\underbrace{\left| \mathbb{E}\Big[f\big(X_T\big) \Big] - \frac{1}{N^2} \sum_{m=1}^{N^2} f\big(Y_N^{N,m}\big) \right|}_{\text{error of the Monte Carlo Euler method} \longrightarrow 0}$$

$$\leq \underbrace{\left| \mathbb{E}\Big[f\big(X_T\big) \Big] - \mathbb{E}\Big[f\big(Y_N^N\big) \Big] \right|}_{\text{time discretization error} \longrightarrow \infty} + \underbrace{\left| \mathbb{E}\Big[f\big(Y_N^N\big) \Big] - \frac{1}{N^2} \sum_{m=1}^{N^2} f\big(Y_N^{N,m}\big) \right|}_{\text{statistical error} \longrightarrow \infty}$$

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Conclusion

Strong and numerically weak error estimates are convenient since stochastic calculus is an L^2-calculus (Itô isometry, etc.).

But, if Euler's method is used to solve one of the nonlinear problems above, then one needs different concepts such as

$$\left| X_T - Y_N^N \right| \xrightarrow{N \to \infty} 0 \qquad \mathbb{P}\text{-a.s.}$$

for the strong approximation problem (Gyöngy (1998)) and

$$\left| \mathbb{E}\Big[f(X_T) \Big] - \frac{1}{N^2} \left(\sum_{m=1}^{N^2} f(Y_N^N) \right) \right| \xrightarrow{N \to \infty} 0 \qquad \mathbb{P}\text{-a.s.}$$

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References

- Hutzenthaler and J (2009), Non-globally Lipschitz Counterexamples for the stochastic Euler scheme.
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