Recitation problems for Tuesday Mar 30, 11:25am-12:15pm

1. Alon and Spencer p.62 #5 (p.57 in 2nd ed.); Hard: #2, #3 on the same page. (1 problem = 1 recitation credit of your needed 3.)

2. Let Q_n be the set of binary strings of length n. Define $w(x)$ to be the number of 1's in a binary string $x \in Q_n$. Let x be a string chosen uniformly at random from Q_n. Use a Chernoff bound in Appendix A of Alon and Spencer to find bounds $L(n)$ and $U(n)$ for which
$$\Pr[L(n) \leq w(x) \leq U(n)] \geq 1 - n^{-2}.$$
Preferred, but not required, is a bound of the form
$$\Pr[L_\alpha(n) \leq w(x) \leq U_\alpha(n)] \geq 1 - n^{-\alpha},$$
For any $\alpha > 0$.

3. An *isolated triangle* in G is a triangle with no other edge connecting one of its three vertices to a vertex not in the triangle. Let X be the number of triangles in $G_{n,p}$. When c is a constant, show that $\mathrm{Var}[X] = E[X](1 + o(1))$. Describe similarities between X and a Poisson random variable (see Wikipedia).

4. Prove the exercise from Tuesday 3/23 on Karp's partitioning algorithm for the Traveling Salesman Problem: Let T^* be an optimal tour. Let C_i be one of the m^2 cells, where C_i is a square with side length $1/m$. Let ℓ_i^* be $T^* \cap C_i$, which is the path segments of the optimal tour that lie inside cell C_i. (Assume that T^* never travels along the boundary of C_i.) Prove that ℓ_i^* can be patched together with new path segments of combined length at most $4/m$ to obtain a tour of the points that lie inside C_i.

5. Prove the details of the asymptotic lower bound of $R(k,k) > \frac{\sqrt{2}}{e}(1 + o(1))k2^{k/2}$ obtained from the bound
$$e\left(\binom{k}{2}\binom{n}{k-2} + 1\right) \cdot 2^{1-\binom{k}{2}} < 1 \quad \Rightarrow \quad R(k,k) > n$$
given by application of the Lovász Local Lemma.

6. (Medium) Compute the expected value of z^* in the improved matching algorithm presented in lecture 10. Convincing evidence presented with a computer algebra system is acceptable. The derivation may include some subtle conditional probabilities. Be sure to make a clean presentation.

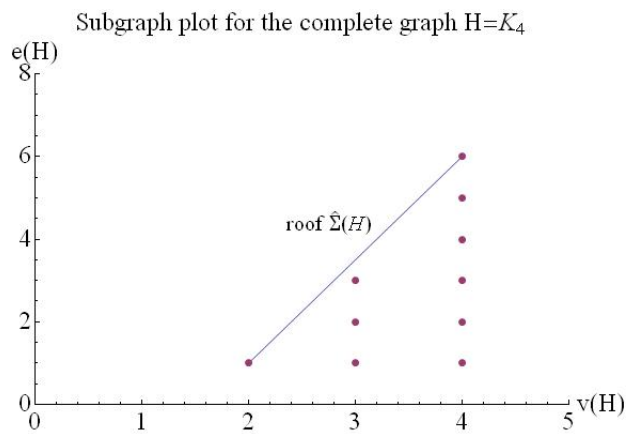
7. Let G be a simple graph. Define $b(G)$ to be the maximum number of edges in a bipartite subgraph of G. For a fixed $v \in V(G)$, recall that $N(v)$ is the neighborhood of v. Each $N(v)$ yields a bipartite subgraph of G in a natural way (for you to determine). Use the Cauchy-Schwarz inequality to show that
$$b(G) \geq \frac{4e^2}{n^2} - \frac{6t}{n}.$$
Hints: every triangle can be viewed as coming from the neighborhood of any one of its vertices; try averaging over the vertices.

8. Apply the alteration/deletion method to a question pertaining to your own research. You must convince the audience that the question you address is reasonable, but do not necessarily have to do a complex calculation.

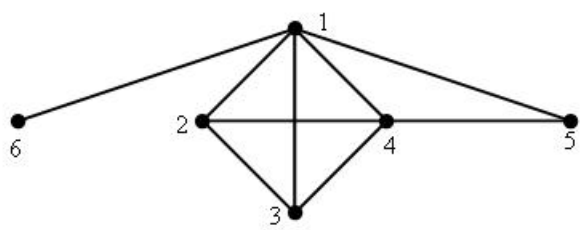
Homework for Thu 4/1/10

1. On Tuesday 3/23 we found a bound on the expected running time of Karp's partitioning algorithm. But we also would like to have a guarantee on the probability of the actual running time being close to the expected running time. Use either a variance calculation or the second moment method to obtain a bound on the probability that the actual running time is close to the expected running time.

2. Read Section 2.2 (starting on page 5) of the paper "Variance of the subgraph count for sparse Erdős-Rényi graphs" in order to understand the subgraph plot of a graph. For example, the subgraph plot of the complete graph $H = K_4$ is as follows.



Now let H be the subgraph whose drawing is given below.



Consider H as a subgraph in the Erdős-Rényi random graph model $G(n,p)$. Draw the subgraph plot for H. For each subgraph H' of H, determine the range of p for which H' contributes nontrivially to the asymptotic variance of the number of copies of H in $G \sim G(n,p)$ by looking at those H' plotted on the roof $\hat{\Sigma}(H)$ of H.

3. **Project Outline.** Project presentations will occur in the 2 to 3 class periods before final exams week. My expectations for time to spend on the project is 10-15 hours, with more time recommended if the project materially contributes to your personal research or the research of your lab or advisor. This is to accommodate the remaining homeworks and recitation questions through the end of the term.

   Turn in the following outline for your project:
   I. Title of project
   II. 100-200 word background information
   III. Describe how you will apply probabilistic methods or use ideas related to this course to your question.
   IV. One or more references, preferably with a web link to a pdf file of the paper.