Homework 3

Recitation problems for Tuesday Feb 9, 11:15am-12:15pm

1. Background: A triangle in a graph is a set of three vertices $\{u, v, w\}$ with all three edges between them present in the graph. Two distinct triangles edge-intersect when the intersection of the two triangles is exactly one edge.)
   Let G be a simple graph with e edges and t triangles. Use the Cauchy-Schwarz inequality (see, for example, Wikipedia and specialize $\mathbf{y} = \mathbf{1}$) to show that there exists a triangle that edge-intersects at least $9t/e - 3$ other triangles.

2. Let G be a simple graph. Define $b(G)$ to be the maximum number of edges in a bipartite subgraph of G. For a fixed $v \in V(G)$, recall that $N(v)$ is the neighborhood of v. Each $N(v)$ yields a bipartite subgraph of G in a natural way (for you to determine). Use the Cauchy-Schwarz inequality as in 1. to show that
$$b(G) \geq \frac{4e^2}{n^2} - \frac{6t}{n}.$$

3. Recall that we proved the existence of triangle-free graphs with arbitrarily large chromatic number. This result can be extended as follows. The *girth* of a graph G is the smallest positive integer g such that the cycle C_g is a subgraph of G. Now let $g \geq 3$ and $k \in \mathbb{Z}^+$, and show there exists a graph with girth $\geq g$ and chromatic number $\geq k$.

4. Demonstrate a nontrivial improvement in the probability that $G(n, 1/2)$ has a perfect matching, from the result we discussed for which the probability was $> 1/3$. For example, showing the probability is at least $1/2 + o(1)$ (as $n \to \infty$) would be acceptable.

Or, any of the previously assigned questions from Alon and Spencer, pp.10-11 (11-12 in 2nd ed.): 3-7, 10. (**Not** #9, which was solved.)

Written problems for Tuesday, Feb 16th, 2010

Turn in at the beginning of class, or in my mailbox at any time before class. Don't forget to read the homework collaboration and use of references policy on the first day handout if you have not already.

1. In a basic sensor network model, n sensors are placed independently uniformly at random inside a unit circle, which has radius 1. All distance are Euclidean (ℓ_2-metric). Each sensor has communication radius λ; assume that $\lambda = o(1)$. Now, approximate the unit circle using a hexagonal lattice as follows.
   (a) Compute the maximum size of the hexagon subject to the constraint that sensors anywhere in two adjacent hexagons are within each other's communication radius λ.
   (b) Overlay the hexagonal lattice on top of the unit circle, and discard all hexagons that are either outside the circle or intersecting the boundary of the circle. Compute the number of remaining hexagons up to the first-order term (correct up to $1 + o(1)$ factor).
   (c) Make the (minor) assumption that all sensors fall inside of a hexagon in the remaining hexagonal lattice. (There is nothing to answer in this part.)
   (d) Let $n \to \infty$, and compute to first order the minimum λ such that with probability $1 - o(1)$, every hexagon contains a sensor.
   (e) Assuming every hexagon contains a sensor, the communication hop-distance between two sensors is at most the length of a shortest path of hexagons between them. To the first-order term, what is the upper bound on the maximum communication hop-distance between any two sensors in the hexagonal lattice assuming λ is given in part (d)?
   (Further thought, not to turn in. What happens when distance is computed with ℓ_p-metric, for

$1 \le p < \infty$? What other structures can be overlaid on the unit circle so that ≥ 1 sensor in each region leads to short communication hop-distances? Are there clear choices for certain values of p?)

2. Alon and Spencer p.23, Question 7 (p.21 in 2nd edition).

3. Show the details proving that the upper bound in Theorem 2.4.1 is a corollary of Theorem 2.4.2.

4. Let G be a simple graph with n vertices $\{v_1, \ldots, v_n\}$ and e edges. Consider the following random experiment to find a bipartite subgraph of G:

   Step 1. Let $t = \lceil \log_2(n+1) \rceil$.
   Let $S(1), \ldots, S(n)$ be pairwise distinct, nonempty subsets of $\{1, \ldots, t\}$.
   Step 2. Let $b_1, \ldots, b_t$ be uniformly and independently chosen bits, from $\{0, 1\}$.
   Step 3. Let $V_0 = \{v_i \in V(G) : \oplus_{j \in S(i)} b_j = 0\}$. ($\oplus$ means addition mod 2.)
   Let $V_1 = \{v_i \in V(G) : \oplus_{j \in S(i)} b_j = 1\}$.
   Output. The bipartite subgraph H of G with parts V_0, V_1, and all edges with exactly one endpoint in each part.

   (a) Compute the expected number of edges in H.
   (b) Compute the number of sample points in the sample space of the experiment. What are the implications of your answer in terms of the expense of doing a naive exhaustive search for a bipartite subgraph with a large number of edges? Compare this to the random experiment in the proof of Theorem 2.2.1 of Alon and Spencer (both editions), and determine if one of the corresponding naive exhaustive searches is better.