PRINT Last name:____________________ First name:____________________

Signature:____________________ Student ID:____________________

Midterm Questions

Quote any formulas you use by listing the Theorem or Corollary number at the place where you use it. Show all major steps.

1. (Erdős-Ko-Rado Theorem.) A family $\mathcal{F}$ of k-subsets of $\{1, \ldots, n\}$ is *intersecting* provided $F_1 \cap F_2 \neq \emptyset$ for all $F_1, F_2 \in \mathcal{F}$. Prove that the maximum size of $\mathcal{F}$ is $\binom{n-1}{k-1}$, and give an explicit family $\mathcal{F}$ achieving this bound.

2. Let G be a graph chosen from the Erdős-Rényi random graph distribution $G(n,p)$; that is, $G \sim G(n,p)$. Define
$$p = c \cdot \frac{\ln n}{n}.$$
(i) Prove that if $c > 1$, then almost always, the number of isolated vertices of G is zero.
(ii) Prove that if $0 < c < 1$, then almost always, the number of isolated vertices of G is concentrated at n^{1-c} (in other words, $\sim n^{1-c}$).

   (Hints: Use the first moment method for one part, and the second moment method for the other part. Remember that $(1 \pm 1/x)^y = \exp(\pm y/x)(1 + o(1))$ for $x, y \to \infty$. Be careful to select an appropriate formula.)

3. **Background.** A hamming ball of radius 1 in the discrete hypercube $\{0,1\}^n$ centered at $u \in \{0,1\}^n$ is the set
$$B(u,1) := \{v \in \{0,1\}^n : d(u,v) \leq 1\}.$$
In other words, $B(u,1)$ consists of u and all length n binary strings that can be obtained from u by flipping 1 bit. For example, If $n = 3$, then $B(000,1) = \{000, 100, 010, 001\}$. Now let $\mathcal{B} = \{B_1, B_2, \ldots, B_m\}$ be a collection of radius 1 hamming balls in $\{0,1\}$. We say that $\mathcal{B}$ is a *covering* provided
$$B_1 \cup B_2 \cup \cdots \cup B_m = \{0,1\}^n;$$
in other words, provided every length n string is an element of at least one hamming ball in $\mathcal{B}$. For example, if $n = 2$, then $\{B(00,1), B(11,1)\}$ is a covering of $\{0,1\}^2$.
Question: Use the deletion/alteration method to select a candidate for a covering of the hypercube $\{0,1\}^n$ with radius 1 hamming balls randomly, and then fix (alter) it so that it is in fact a covering. Optimize the parameter involved in your random construction to find an upper bound on the smallest possible size of a cover.