SHOW WORK FOR FULL CREDIT NO CALCULATORS

1. Peter, Paul, and Mary each flip identical biased coins, and they each stop flipping the first time their own coin is heads. The number of flips for Peter, Paul, and Mary, are Y_1, Y_2, and Y_3, respectively, each independent and having geometric distribution with mean 6. Define $U = Y_1 + Y_2 + Y_3$.
(a) Compute $E(U)$ by finding the moment generating function for U in terms of those of Y_1, Y_2, and Y_3, and then identifying the mean of the resulting distribution.
(b) Compute $E(U)$ using properties of expected value.

(a) $m_{Y_i}(t) = \dfrac{pe^t}{1-(1-p)e^t}$ with $\frac{1}{p} = E(Y_i) = 6 \Rightarrow p = \frac{1}{6}$.

$m_U(t) = E(e^{tU}) = E(e^{t(Y_1+Y_2+Y_3)}) = E(e^{tY_1}e^{tY_2}e^{tY_3}) \underset{\text{indep-endence}}{\overset{\text{by}}{=}} \prod_{i=1}^{3} m_{Y_i}(t)$

$= \left[\dfrac{pe^t}{1-(1-p)e^t}\right]^3.$

(b) $E(U) = E(Y_1) + E(Y_2) + E(Y_3) = 3 \cdot 6 = 18$

Thus U is negative binomial with $r=3$, and $E(U) = \frac{r}{p} = \frac{3}{1/6} = \boxed{18}$

2. Y_1 and Y_2 are discrete random variables whose joint distribution $p(y_1, y_2)$ is given in the table.
(a) Determine the marginal probability functions $p_1(y_1)$ and $p_2(y_2)$.
(b) Compute $P(Y_1 = 3 \mid Y_2 = 1)$.
(c) Are Y_1 and Y_2 independent? If yes, explain why. If not, re-define $p(y_1, y_2)$ so that Y_1 and Y_2 are independent but have the same marginal probability functions as in **(a)**.

	$p(y_1, y_2)$	Y_1: 1	2	3		
Y_2	1	.12	.12	.06	.3	$p_2(1)$
	2	.28	.18	.24	.7	$p_2(2)$
		.4	.3	.3		
		$p_1(1)$	$p_1(2)$	$p_1(3)$		← (a)

(b) $P(Y_1 = 3 \mid Y_2 = 1) = \dfrac{p(3,1)}{p_2(1)} = \dfrac{.06}{.3} = \boxed{\frac{1}{5}}$

(c) No. $p(2,1) = .12 \neq p_1(2)p_2(1) = (.3)(.3) = .09$.

to make Y_1, Y_2 independent with same marginals, need $p(y_1, y_2) = p_1(y_1)p_2(y_2)$ for all y_1, y_2.

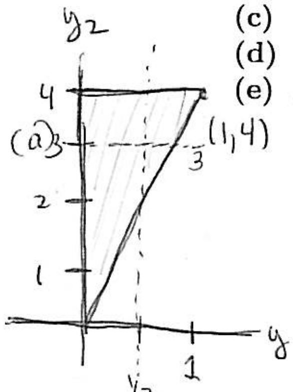
redefine $p(2,1) = .09$ $p(3,1) = .09$

$p(2,2) = .21$ $p(3,2) = .21$

3. (16pts) Let Y_1 and Y_2 be continuous random variables with joint density

$$f(y_1, y_2) = \begin{cases} \frac{1}{2}, & 0 \le y_2 \le 4, \quad 0 \le y_1 \le 1, \quad 4y_1 \le y_2 \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Sketch the region of support of $f(y_1, y_2)$ (where $f(y_1, y_2) > 0$).
(b) Compute the marginal probability densities $f_1(y_1)$ and $f_2(y_2)$.
(c) Compute the conditional probability density of Y_1 given that $Y_2 = 1$.
(d) Compute $P(Y_1 \le 1/2 \mid Y_2 \le 3)$.
(e) Are Y_1 and Y_2 independent? Briefly explain why or why not.



(b) $f_1(y_1) = \int_{4y_1}^{4} \frac{1}{2}\, dy_2 = \frac{y_2}{2}\Big|_{4y_1}^{4} = \frac{4}{2} - 2y_1 = \boxed{2-2y_1}$

$f_2(y_2) = \int_0^{y_2/4} \frac{1}{2}\, dy_1 = \frac{y_1}{2}\Big|_0^{y_2/4} = \boxed{\frac{y_2}{8}}$

(c) $f(y_1 | 1) = \frac{f(y_1, 1)}{f_2(1)} = \frac{\frac{1}{2}}{\frac{1}{8}} = \boxed{4}, \; 0 \le y_1 \le \frac{1}{4}$

(d) $P(Y_1 \le \frac{1}{2} | Y_2 \le 3) = \frac{P(Y_1 \le \frac{1}{2}, Y_2 \le 3)}{P(Y_2 \le 3)} = \frac{\int_0^{\frac{1}{2}} \int_{4y_1}^{3} \frac{1}{2}\, dy_2\, dy_1}{\int_0^3 f_2(y_2)\, dy_2} = \frac{\int_0^{\frac{1}{2}} \frac{y_2}{2}\Big|_{4y_1}^{3} dy_1}{\int_0^3 \frac{y_2}{8}\, dy_2}$

$= \frac{\int_0^{\frac{1}{2}} (\frac{3}{2} - 2y_1)\, dy_1}{\frac{y_2^2}{16}\Big|_0^3} = \frac{\frac{3}{2}y_1 - y_1^2 \Big|_0^{\frac{1}{2}}}{\frac{9}{16}} = \frac{\frac{3}{4} - \frac{1}{4}}{\frac{9}{16}} = \frac{\frac{1}{2}}{\frac{9}{16}} = \frac{16}{18} = \boxed{\frac{8}{9}}$

(e) No. the support of $f(y_1, y_2)$ is not a rectangle

4. Assume that X and U are random variables with

$$E(X) = -2 \quad V(X) = 3$$
$$E(U) = 1 \quad V(U) = 5$$
$$E(XY) = 1\,.$$

(a) Compute $E(4X - 2U)$.
(b) Compute $V(4X - 2U)$.
(c) Are X and U independent? Briefly explain why or why not.

(a) $E(4X - 2U) = 4E(X) - 2E(U) = -8 - 2 = \boxed{-10}$

(b) $V(4X - 2U) = 16V(X) + (-2)^2 V(U) + 2(4)(-2)\,Cov(X, U)$

$Cov(X, U) = E(XU) - E(X)E(U) = 1 - (-2)(1) = 3$

so $V(4X - 2U) = 16(3) + 4(5) - 16(3) = \boxed{20}$

5. On a typical day, a store sells a fraction Y_1 of its stock of milk and a fraction Y_2 of its stock of flour. The store's revenue from these sales is $R = 400Y_1 + 100Y_2$. The joint density function for Y_1 and Y_2 is

$$f(y_1, y_2) = \begin{cases} y_1 + y_2, & 0 \le y_1 \le 1, \quad 0 \le y_2 \le 1 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Compute $E(R)$.
(b) Are Y_1 and Y_2 independent? Briefly explain why or why not.

(a) $E(Y_1) = \int_0^1\int_0^1 (y_1^2 + y_1 y_2)\,dy_2\,dy_1 = \int_0^1 \left(y_1^2 y_2 + \frac{y_1 y_2^2}{2}\right)\Big|_0^1 dy_1 = \int_0^1 \left(y_1^2 + \frac{y_1}{2}\right) dy_1$

$= \frac{y_1^3}{3} + \frac{y_1^2}{4}\Big|_0^1 = \frac{1}{3} + \frac{1}{4} = \boxed{\frac{7}{12}}$ (3)

By symmetry of y_1 and y_2, $E(Y_2) = \boxed{\frac{7}{12}}$ (3)

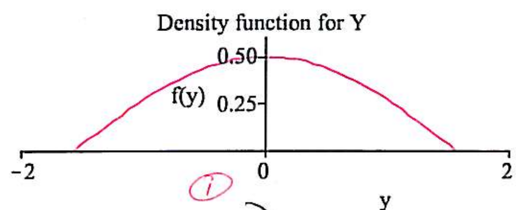
(3)

So $E(R) = E(400Y_1 + 100Y_2) = 400E(Y_1) + 100E(Y_2) = \frac{500 \cdot 7}{12} = \frac{3500}{12} = \boxed{\frac{875}{3}}$

(b) $\boxed{\text{No.}}$ $f(y_1, y_2)$ cannot factor into $f_1(y_1)\cdot f_2(y_2)$ even though the support of $f(y_1, y_2)$ is a rectangle. (3)

6. Assume Y is a continuous random variable with joint density function

$$f_Y(y) = \begin{cases} \frac{1}{2}\cos(y), & -\pi/2 \le y \le \pi/2 \\ 0, & \text{otherwise.} \end{cases}$$



Let $U = Y^2$. (Not 1-1 on the support of Y.)
(a) Compute the density $f_U(u)$ of U.
(b) Give the interval on which U is supported.

(a)

(2) (1)

$F_U(u) = P(U \le u) = P(Y^2 \le u) = P(-\sqrt{u} \le Y \le \sqrt{u})$

(1) (1) (1)

$= \int_{-\sqrt{u}}^{\sqrt{u}} \frac{\cos y}{2}\,dy = \frac{\sin y}{2}\Big|_{-\sqrt{u}}^{\sqrt{u}} = \frac{\sin\sqrt{u}}{2} - \frac{\sin(-\sqrt{u})}{2} = \sin\sqrt{u}.$ (it is also possible to find $F_Y(y)$ first)

(2) (1)

$f_U(u) = \frac{d}{du}F_U(u) = \frac{d}{du}\sin\sqrt{u} = \boxed{(\cos\sqrt{u})\frac{1}{2\sqrt{u}}}$

(b) since $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$, (3)

$\boxed{0 \le u \le \frac{\pi^2}{4}}$

(we might exclude 0 since $f_U(0)$ is undefined)

7. Let Y be a continuous random variable with density function

$$f_Y(y) = \begin{cases} \frac{5}{y^2}, & y > 5 \\ 0, & \text{otherwise.} \end{cases}$$

Define $U = 3\ln(Y-5)$.

9 **(a)** Compute the density function $f_U(u)$ for U.

3 **(b)** Give the interval of support of U.

(a) via formula, $f_U(u) = f_Y(h^{-1}(u)) \left| \frac{dh^{-1}}{du} \right|$

$h^{-1}(u)$: $u = 3\ln(y-5)$

$\frac{u}{3} = \ln(y-5)$

$e^{u/3} = y-5$

$5+e^{u/3} = y$

$h^{-1}(u) = 5+e^{u/3}$

$\frac{dh^{-1}(u)}{du} = \frac{1}{3}e^{u/3}$

so $f_U(u) = \frac{5}{(5+e^{u/3})^2} \cdot \frac{1}{3}e^{u/3} = \boxed{\frac{5e^{u/3}}{3(5+e^{u/3})^2}}$

(b) $\infty > y > 5$

$\infty > y-5 > 0$

$\infty > \ln(y-5) > -\infty$

$\infty > 3\ln(y-5) > -\infty$

$\boxed{\text{support of } U \text{ is } (-\infty, \infty)}$

8. Let $Y_1, Y_2, \ldots, Y_{16}$ be independent random variables each with $E(Y_i) = \mu$ and $V(Y_i) = \sigma^2$. They represent 16 repeated independent samples from the same population. The function

$$\overline{Y} := \frac{Y_1 + Y_2 + \cdots + Y_{16}}{16}$$

is called the *sample mean* of the population. Use Tschebycheff's Theorem to find the probability that $\overline{Y}$ is within $\sigma/3$ of $E(\overline{Y})$.

3 $E(\overline{Y}) = E\left(\frac{\sum_{i=1}^{16} Y_i}{16}\right) = \frac{1}{16}\sum_{i=1}^{16} E(Y_i) = \frac{1}{16} \cdot 16 \cdot \mu = \mu.$

4 $V(\overline{Y}) = V\left(\sum_{i=1}^{16} \frac{Y_i}{16}\right) = \sum_{i=1}^{16} V\left(\frac{Y_i}{16}\right) + \underbrace{\sum_{1 \le i < j \le 16} 2\,\mathrm{Cov}\left(\frac{Y_i}{16}, \frac{Y_j}{16}\right)}_{0 \text{ by independence}}$ 1

$= \sum_{i=1}^{16} \frac{1}{16^2} V(Y_i) = 16 \cdot \frac{1}{16^2}\sigma^2 = \frac{\sigma^2}{16} = \sigma_{\overline{Y}}^2$ 1

1 $P(|\overline{Y} - E(\overline{Y})| \ge \sigma/3) = P(|\overline{Y} - \mu| \ge \sigma/3) = P(|\overline{Y} - \mu| \ge \frac{4}{3}\sigma_{\overline{Y}}) \le \frac{1}{(4/3)^2}$ 1 1

$= \boxed{\frac{9}{16}}$ 1

set $= k\sigma_{\overline{Y}}$ 1

$\frac{\sigma_{Y_i}}{3} = k\sigma_{\overline{Y}}$

$\frac{\sigma}{3} = k\frac{\sigma}{4}$

$\frac{4}{3} = k$