SHOW WORK FOR FULL CREDIT NO CALCULATORS

1. On a typical day, a store sells a fraction Y_1 of its stock of milk and a fraction Y_2 of its stock of soda. The store's revenue from these sales is $R = 300Y_1 + 200Y_2$. The joint density function for Y_1 and Y_2 is

$$f(y_1, y_2) = \begin{cases} 1 + y_1 - y_2, & 0 \le y_1 \le 1, \quad 0 \le y_2 \le 1 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Compute $E(R)$.
(b) Are Y_1 and Y_2 independent? Briefly explain why or why not.

(a)

$$E(Y_1) = \int_0^1\int_0^1 (y_1 + y_1^2 - y_1y_2)\,dy_2\,dy_1 = \int_0^1 \left(y_1y_2 + y_1^2y_2 - \frac{y_1y_2^2}{2}\right)\Big|_0^1 dy_1$$

$$= \int_0^1 \left(y_1 + y_1^2 - \frac{y_1}{2}\right) dy_1 = \frac{y_1^2}{2} + \frac{y_1^3}{3} - \frac{y_1^2}{4}\Big|_0^1 = \frac{1}{2} + \frac{1}{3} - \frac{1}{4} = \frac{6+4-3}{12} = \frac{7}{12}$$

$$E(Y_2) = \int_0^1\int_0^1 (y_2 + y_1y_2 - y_2^2)\,dy_1\,dy_2 = \int_0^1 \left(y_2y_1 + \frac{y_1^2y_2}{2} - y_1y_2^2\right)\Big|_0^1 dy_2$$

$$= \int_0^1 \left(y_2 + \frac{y_2}{2} - y_2^2\right) dy_2 = \frac{y_2^2}{2} + \frac{y_2^2}{4} - \frac{y_2^3}{3}\Big|_0^1 = \frac{1}{2} + \frac{1}{4} - \frac{1}{3} = \frac{6+3-4}{12} = \frac{5}{12}$$

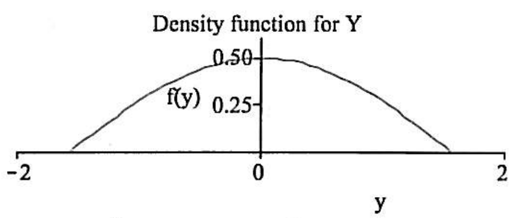
$$E(R) = E(300Y_1 + 200Y_2) = 300E(Y_1) + 200E(Y_2) = \frac{300\cdot 7}{12} + \frac{200\cdot 5}{12}$$

$$= 175 + \frac{250}{3} = \boxed{\frac{775}{3}}$$

(b) No. $f(y_1,y_2)$ does not factor into any $f_1(y_1)\cdot f_2(y_2)$ even though its support is on a rectangle.

2. Assume Y is a continuous random variable with joint density function

$$f_Y(y) = \begin{cases} \frac{1}{2}\cos(y), & -\pi/2 \le y \le \pi/2 \\ 0, & \text{otherwise.} \end{cases}$$



Let $U = Y^2$. (Not 1-1 on the support of Y.)
(a) Compute the density $f_U(u)$ of U.
(b) Give the interval on which U is supported.

(a)

$$F_U(u) = P(U \le u) = P(Y^2 \le u) = P(-\sqrt{u} \le Y \le \sqrt{u}) = \int_{-\sqrt{u}}^{\sqrt{u}} \frac{\cos y}{2}\,dy$$

$$= \frac{\sin y}{2}\Big|_{-\sqrt{u}}^{\sqrt{u}} = \frac{\sin\sqrt{u}}{2} - \frac{\sin(-\sqrt{u})}{2} = \sin\sqrt{u}.$$

$$f_U(u) = \frac{d}{du}F_U(u) = (\cos\sqrt{u})\frac{1}{2\sqrt{u}}$$

(b) Since $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$, $\boxed{0 \le u \le \frac{\pi^2}{4}}$

[we might exclude 0 since $f_U(0)$ is undefined]

3. Let Y be a continuous random variable with density function

$$f_Y(y) = \begin{cases} \frac{3}{y^2}, & y > 3 \\ 0, & \text{otherwise.} \end{cases}$$

Define $U = 5\ln(Y-3)$.
(a) Compute the density function $f_U(u)$ for U.
(b) Give the interval of support of U.

(a) Via formula, $f_U(u) = f_Y(h^{-1}(u))\left|\frac{dh^{-1}}{du}\right|$

$h^{-1}(u)$: $u = 5\ln(y-3)$

$\frac{u}{5} = \ln(y-3)$

$e^{u/5} = y-3$

$3+e^{u/5} = y$

$h^{-1}(u) = 3+e^{u/5}$

$\frac{d}{du}h^{-1}(u) = \frac{e^{u/5}}{5}$ so $f_U(u) = \frac{3}{(3+e^{u/5})^2}\frac{e^{u/5}}{5} = \boxed{\frac{3}{5}\frac{e^{u/5}}{(3+e^{u/5})^2}}$

(b) $3 < y < \infty$

$0 < y-3 < \infty$

$-\infty < \ln(y-3) < \infty$

$-\infty < 5\ln(y-3) < \infty$

$\boxed{\text{support of } U \text{ is } (-\infty, \infty)}$.

4. Let $Y_1, Y_2, \ldots, Y_{25}$ be independent random variables each with $E(Y_i) = \mu$ and $V(Y_i) = \sigma^2$. They represent 25 repeated independent samples from the same population. The function

$$\overline{Y} := \frac{Y_1 + Y_2 + \cdots + Y_{25}}{25}$$

is called the *sample mean* of the population. Use Tschebycheff's Theorem to find the probability that $\overline{Y}$ is within $\sigma/2$ of $E(\overline{Y})$.

$$E(\overline{Y}) = E\left(\frac{\sum_{i=1}^{25} Y_i}{25}\right) = \frac{1}{25}\sum_{i=1}^{25} E(Y_i) = \frac{1}{25}\cdot 25\cdot\mu = \mu.$$

$$V(\overline{Y}) = V\left(\sum_{i=1}^{25}\frac{Y_i}{25}\right) = \sum_{i=1}^{25} V\left(\frac{Y_i}{25}\right) + \underbrace{\sum_{1\le i<j\le 25} Cov\left(\frac{Y_i}{25}, \frac{Y_j}{25}\right)}_{\to 0 \text{ by independence}}$$

$$= 25\cdot\left(\frac{1}{25}\right)^2 V(Y_i) = \frac{\sigma^2}{25} = \left(\frac{\sigma}{5}\right)^2 = \sigma_{\overline{Y}}^2$$

$$P(|\overline{Y} - E(\overline{Y})| \ge \sigma/2) = P(|\overline{Y}-\mu| \ge \underbrace{\sigma/2}_{\text{set} = k\sigma_{\overline{Y}}}) = P(|\overline{Y}-\mu| \ge \tfrac{5}{2}\sigma_{\overline{Y}}) \le \frac{1}{(5/2)^2} = \boxed{\frac{4}{25}}$$

$\frac{\sigma}{2} = k\sigma_{\overline{Y}}$

$\frac{\sigma}{2} = k\frac{\sigma}{5}$

$\frac{5}{2} = k$

5. Three radioactive atoms are observed until they decay. The individual times of decay are Y_1, Y_2, and Y_3, respectively, which are independent and exponentially distributed, each with mean 12 (seconds). Define $U = Y_1 + Y_2 + Y_3$.
(a) Compute $E(U)$ by finding the moment generating function for U in terms of those of Y_1, Y_2, and Y_3, and then identifying the mean of the resulting distribution.
(b) Compute $E(U)$ using properties of expected value.

(a) $m_{Y_i}(t) = (1-12t)^{-1}$ 2

$m_U(t) = E(e^{tU}) = E(e^{t(Y_1+Y_2+Y_3)}) = E(e^{tY_1}e^{tY_2}e^{tY_3}) = \prod_{i=1}^{3} m_{Y_i}(t) = \boxed{(1-12t)^{-3}}$ 2 2 1

by independence

Thus U is gamma distributed with mean $(+3)(12) = \boxed{36}$ 2

3 (b) $E(U) = E(Y_1+Y_2+Y_3) = E(Y_1)+E(Y_2)+E(Y_3)$

$= 12+12+12 = \boxed{36}$

6. Y_1 and Y_2 are discrete random variables whose joint distribution $p(y_1, y_2)$ is given in the table.
4 **(a)** Determine the marginal probability functions $p_1(y_1)$ and $p_2(y_2)$.
4 **(b)** Compute $P(Y_1 \geq 2 \mid Y_2 = 1)$.
4 **(c)** Are Y_1 and Y_2 independent? If yes, explain why. If not, re-define $p(y_1, y_2)$ so that Y_1 and Y_2 are independent but have the same marginal probability functions as in **(a)**.

	$p(y_1, y_2)$	Y_1 = 1	2	3	$p_2(y_2)$
Y_2	1	.1	.2	.1	.4
	2	.2	.3	.1	.6
	$p_1(y_1)$	.3	.5	.2	

(a)

(b) $P(Y_1 \geq 2 \mid Y_2 = 1) = \dfrac{P(Y_1 \geq 2, Y_2 = 1)}{P(Y_2 = 1)}$

$= \dfrac{.2+.1}{.4} = \boxed{\dfrac{3}{4}}$

(c) no. $p(1,1) \neq p_1(1)p_2(1)$.

redefine table:

$p(1,1) = .12$ $p(3,1) = .08$

$p(1,2) = .18$ $p(3,2) = .12$

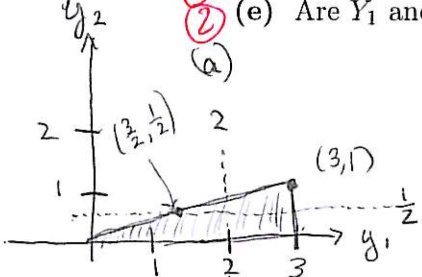
$\frac{3}{25}$	$\frac{1}{5}$	$\frac{2}{25}$
$\frac{9}{50}$	$\frac{3}{10}$	$\frac{3}{25}$

7. (16pts) Let Y_1 and Y_2 be continuous random variables with joint density

$$f(y_1, y_2) = \begin{cases} \frac{2}{3}, & 0 \le y_1 \le 3, \quad 0 \le y_2 \le 1, \quad 3y_2 \le y_1 \\ 0, & \text{elsewhere.} \end{cases}$$

2 **(a)** Sketch the region of support of $f(y_1, y_2)$ (where $f(y_1, y_2) > 0$).
4 **(b)** Compute the marginal probability densities $f_1(y_1)$ and $f_2(y_2)$.
4 **(c)** Compute the conditional probability density of Y_1 given that $Y_2 = \frac{1}{3}$.
4 **(d)** Compute $P(Y_2 \le 1/2 \,|\, Y_1 \le 2)$.
2 **(e)** Are Y_1 and Y_2 independent? Briefly explain why or why not.

(a)



(b) $f_1(y_1) = \int_0^{y_1/3} \frac{2}{3}\, dy_2 = \frac{2}{3} y_2 \Big|_0^{y_1/3} = \boxed{\frac{2y_1}{9}}, \quad 0 \le y_1 \le 3$

$f_2(y_2) = \int_{3y_2}^{3} \frac{2}{3}\, dy_1 = \frac{2}{3} y_1 \Big|_{3y_2}^{3} = \boxed{2 - 2y_2}, \quad 0 \le y_2 \le 1$

(c) $f(y_1 | \frac{1}{3}) = \dfrac{f(y_1, \frac{1}{3})}{f_2(\frac{1}{3})} = \dfrac{2/3}{4/3} = \boxed{\frac{1}{2}} \quad 1 \le y_1 \le 3$

(d) $P(Y_2 \le \frac{1}{2} | Y_1 \le 2) = \dfrac{P(Y_2 \le \frac{1}{2}, Y_1 \le 2)}{P(Y_1 \le 2)} = \dfrac{\int_0^{\frac{1}{2}} \int_{3y_2}^{2} \frac{2}{3}\, dy_1\, dy_2}{\int_0^2 f_1(y_1)\, dy_1} = \dfrac{\int_0^{\frac{1}{2}} \frac{2}{3} y_1 \Big|_{3y_2}^{2} dy_2}{\int_0^2 \frac{2y_1}{9}\, dy_1}$

$= \dfrac{\int_0^{\frac{1}{2}} \left(\frac{4}{3} - 2y_2\right) dy_2}{y_1^2/9 \Big|_0^2} = \dfrac{\frac{4}{3} y_2 - y_2^2 \Big|_0^{\frac{1}{2}}}{\cancel{2}\ 4/9} = \dfrac{\frac{2}{3} - \frac{1}{4}}{4/9} = \dfrac{5/12}{4/9} = \dfrac{45}{48} = \boxed{\frac{15}{16}}$

(e) no, support is not a rectangle.

$\int_{3/2}^{2} \int_0^{1/2} \frac{2}{3}\, dy_2\, dy_1 = (\frac{1}{2})(\frac{1}{2})(\frac{2}{3}) = \frac{1}{6}$

d) alternate:
$P(Y_2 \le \frac{1}{2}, Y_1 \le 2) = \int_0^{3/2} \int_0^{y_1/3} \frac{2}{3}\, dy_2\, dy_1 = \int_0^{3/2} \frac{2}{3} y_2 \Big|_0^{y_1/3} dy_1 = \int_0^{3/2} \frac{2y_1}{9}\, dy_1 = \frac{y_1^2}{9} \Big|_0^{3/2} = \frac{1}{4} + \frac{1}{6} = \frac{10}{24} = \frac{5}{12}$

8. Assume that X and Y are random variables with

$$\begin{array}{ll} E(X) = 3 & V(X) = 4 \\ E(Y) = -1 & V(Y) = 2 \\ E(XY) = -1\,. & \end{array}$$

5 **(a)** Compute $E(3X - 5Y)$.
5 **(b)** Compute $V(3X - 5Y)$.
2 **(c)** Are X and Y independent? Briefly explain why or why not.

(a) $E(3X - 5Y) = 3E(X) - 5E(Y) = 3 \cdot 3 - 5(-1) = \boxed{14}$

(b) $V(3X - 5Y) = V(3X) + V(-5Y) + 2\,\mathrm{Cov}(3X, -5Y)$

$= 9V(X) + 25V(Y) + 2(3)(-5)\,\mathrm{Cov}(X, Y)$ $\quad \mathrm{Cov}(X,Y) = E(XY) - E(X)E(Y)$

$= 36 + 50 + (-30)(-1 - 3(-1))$

$= 86 + (-30)(2) = \boxed{26}$