PRINT Last name: KEY First name: ____________

Signature: ____________ Student ID: ____________

Math 475 Exam 2, Fall 2007

Instructions. You must show work in order to receive full credit. Partial credit is possible for work which makes positive progress toward the solution.

Conditions. ***No calculators***, notes, books, or scratch paper. By writing your name on the exam you certify that all work is your own, under penalty of all remedies outlined in the student handbook. Please do not talk until after leaving the room.

Time limit: 1 hour 15 minutes (strict).

NOTE: The topics may not be in order either of increasing difficulty or of the order they were covered in the course. Answers (sums/products/quotients of fractions or binomial coefficients) **do not need to be fully simplified**, nor do you need to find numerical approximations for your answers.

Distribution	Probability function	Mean	Variance
Normal	$f(y) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[\frac{-(y-\mu)^2}{2\sigma^2}\right]$	μ	σ^2
Hypergeometric	$p(y) = \frac{\binom{r}{y}\binom{N-r}{n-y}}{\binom{N}{n}}$	$\frac{nr}{N}$	$n\left(\frac{r}{N}\right)\left(\frac{N-r}{N}\right)\left(\frac{N-n}{N-1}\right)$
Binomial			
Geometric			$\frac{1-p}{p^2}$
Poisson	$p(y) = \frac{\lambda^y}{y!}e^{-\lambda}$		
Negative binomial	$p(y) = \binom{y-1}{r-1}p^r(1-p)^{y-r}$	$\frac{r}{p}$	$\frac{r(1-p)}{p^2}$
Uniform			

Markov: $P(|Y| \geq \alpha) \leq \frac{E(Y)}{\alpha}$

Tchebysheff: $P(|Y-\mu| \geq k\sigma) \leq \frac{1}{k^2}$

SHOW WORK FOR FULL CREDIT NO CALCULATORS

1. The probability function $p(y)$ for a discrete random variable Y is given in the following table. Compute $\mu = E(Y)$, $\sigma^2 = V(Y)$, and $E(Y^2 - Y + 2)$. For full credit use properties of expected value to reduce the total number of calculations needed to produce all 3 answers.

y	0	1	2	3
$p(y)$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{3}$

$E(Y) = 0\cdot\frac{1}{6} + 1\cdot\frac{1}{3} + 2\cdot\frac{1}{6} + 3\cdot\frac{1}{3} = \frac{2}{6} + \frac{2}{6} + \frac{6}{6} = \frac{10}{6} = \boxed{\frac{5}{3}}$

$E(Y^2) = 0\cdot\frac{1}{6} + 1\cdot\frac{1}{3} + 4\cdot\frac{1}{6} + 9\cdot\frac{1}{3} = \frac{2}{6} + \frac{4}{6} + \frac{18}{6} = \frac{24}{6} = 4$

$V(Y) = E(Y^2) - E(Y)^2 = 4 - \left(\frac{5}{3}\right)^2 = \frac{36}{9} - \frac{25}{9} = \boxed{\frac{11}{9}}$

$E(Y^2 - Y + 2) = E(Y^2) - E(Y) + 2 = 4 - \frac{5}{3} + 2 = \frac{12}{3} - \frac{5}{3} + \frac{6}{3} = \boxed{\frac{13}{3}}$

2. For this problem, let Y be a discrete random variable with binomial distribution with n trials and success probability p.
(a) Write down the formula for the probability function $p(y)$, also giving the values of y for which $p(y) > 0$.
(b) If $n = 3$ and $p = 1/4$, compute $P(Y \leq 1 | Y \leq 2)$. (Full simplification not required.)
(c) Now assume $p = 1/2$. Find the smallest n such that $P(Y \leq 1) \leq 1/6$ by finding a general formula for $P(Y \leq 1)$ in terms of n.

(a) $p(y) = \boxed{\binom{n}{y} p^y (1-p)^{n-y}, \quad y = 0, 1, \ldots, n}$

(b) $n=3, p=\frac{1}{4}$

$p(0) = \left(\frac{3}{4}\right)^3 = \frac{27}{64}$

$p(1) = 3\cdot\frac{1}{4}\left(\frac{3}{4}\right)^2 = \frac{27}{64}$

$p(2) = 3\left(\frac{1}{4}\right)^2\frac{3}{4} = \frac{9}{64}$

$p(3) = \left(\frac{1}{4}\right)^3 = \frac{1}{64}$

$P(Y \leq 1 | Y \leq 2) = \frac{P(Y \leq 1 \text{ and } Y \leq 2)}{P(Y \leq 2)} = \frac{P(Y \leq 1)}{P(Y \leq 2)}$

$= \frac{p(0) + p(1)}{p(0) + p(1) + p(2)} = \frac{\left(\frac{3}{4}\right)^3 + 3\left(\frac{1}{4}\right)\left(\frac{3}{4}\right)^2}{\left(\frac{3}{4}\right)^3 + 3\left(\frac{1}{4}\right)\left(\frac{3}{4}\right)^2 + 3\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)} = \boxed{\frac{6}{7}}$

(c) $P(Y \leq 1) = p(0) + p(1) = \left(\frac{1}{2}\right)^n + \binom{n}{1}\left(\frac{1}{2}\right)^n = \frac{n+1}{2^n}$

n	$\frac{n+1}{2^n}$
1	1
2	3/4
3	4/8
4	5/16
5	6/32 = 1/16

n	$\frac{n+1}{2^n}$
6	$7/64 < \frac{1}{6}$

2^n increases faster than $n+1$ when $n \geq 1$, so

$\boxed{n=6}$ is the smallest such n

3. A (theoretical) random number generator produces a random number Y which is a continuous random variable with uniform distribution on the interval $[5, 12]$. Assume that successive numbers produced are independent. If 20 numbers are produced, how many do we expect to lie in the subinterval $[10, 12]$?

$$P(10 \le Y \le 12) = \int_{10}^{12} f(y)\,dy = \int_{10}^{12} \frac{1}{12-5}\,dy = \frac{2}{7}.$$

Define X to be the number of numbers between 10 and 12 ~~of generating 20~~ generated ~~(independent) numbers~~ by using Y.

Then X is binomial with $n = 20$, $p = \frac{2}{7}$.

$$E(X) = np = \boxed{\frac{40}{7}}$$

4. The standard normal random variable Z satisfies $P(Z \geq 1) \doteq 0.1587$, $P(Z \geq 1.5) \doteq 0.0668$, and $P(Z \geq 2) \doteq 0.0228$. The actual height Y of a randomly selected tree in a large forest is normally distributed with mean 80 and standard deviation 5 (in feet).
(a) Give a transformation expressing $P(Y \geq y)$ in terms of $P(Z \geq z)$.
(b) Based on the given probabilities for Z, compute $P(|Y - 80| \geq c)$ for three distinct values of c.

(a) $\boxed{P(Y \ge y) = P\left(\frac{Y-80}{5} \ge \frac{y-80}{5}\right) = P\left(Z \ge \frac{y-80}{5}\right)}$

since $\frac{Y - E(Y)}{\sigma(Y)}$ is standard normal when Y is normal.

(b) $P(|Y-80| \ge c) = P(-c \ge Y-80 \ge c) = 2P(Y-80 \ge c)$

$= 2P\left(\frac{Y-80}{5} \ge \frac{c}{5}\right)$. need $\frac{c}{5} = 1, 1.5, 2$ or $c = 5, 7.5, 10$.

$P(|Y-80| \ge 5) = 2P(Z \ge 1) \doteq \boxed{2(0.1587)}$

$P(|Y-80| \ge 7.5) = 2P(Z \ge 1.5) \doteq \boxed{2(0.0668)}$

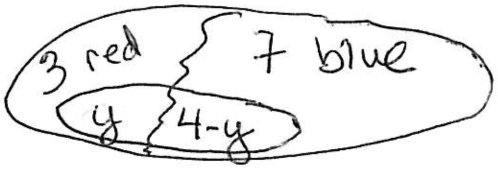
$P(|Y-80| \ge 10) = 2P(Z \ge 2) \doteq \boxed{2(0.0228)}$

5. An urn contains ten balls, of which three are red and seven are blue (otherwise the balls are indistinguishable). Four balls are drawn from the urn *without replacement.* Let Y be the number of red balls drawn.

(a) What is the probability that exactly two of the drawn balls are red?

(b) Determine $E(Y)$ and $V(Y)$.

(a)



$P(Y=2) = p(2)$ for hypergeom. distr.

$$p(2) = \frac{\binom{3}{2}\binom{10-3}{4-2}}{\binom{10}{4}} = \frac{3\cdot 21}{210} = \boxed{\frac{3}{10}}$$

(b) $N=10,\ n=4,\ r=3$

$$E(Y) = \boxed{\frac{12}{10}}$$

$$V(Y) = 4\left(\frac{3}{10}\right)\left(\frac{10-3}{10}\right)\left(\frac{10-4}{10-1}\right)$$

$$= 4\left(\frac{3}{10}\right)\left(\frac{7}{10}\right)\left(\frac{6}{9}\right) = \boxed{\frac{14}{25}}$$

6. Recall that the moment generating function for a discrete random variable Y is defined to be

$$m(t) := E\left(1 + tY + \frac{t^2Y^2}{2!} + \frac{t^3Y^3}{3!} + \cdots\right). \tag{1}$$

Compute $V(Y)$ if the moment generating function for Y is $m(t) = \exp\left(3\left(e^t - 1\right)\right)$. (Hints: It might help to rewrite the right-hand side of (1) in terms of the moments μ_k' of Y. If you recognize this moment generating function, you must still go through the computation, but of course you can check your answer with what you know about the distribution of Y.)

$$m(t) = 1 + tE(Y) + \frac{t^2}{2!}E(Y^2) + \frac{t^3}{3!}E(Y^3) + \cdots = \sum_{k=0}^{\infty} \frac{t^k}{k!}\mu_k'$$

$$E(Y) = \frac{d}{dt}m(t)\Big|_{t=0} = \exp(3(e^t-1))\,3e^t\Big|_{t=0} = \exp(3(1-1))\,3e^0 = \boxed{3}$$

$$E(Y^2) = \frac{d^2}{dt^2}m(t)\Big|_{t=0} = \frac{d}{dt}\left[\exp(3(e^t-1))\,3e^t\right]\Big|_{t=0}$$

$$= \exp(3(e^t-1))\,3e^t\,3e^t + \exp(3(e^t-1))\,3e^t\Big|_{t=0}$$

$$= \exp(3\cdot 0)\,3e^0\,3e^0 + \exp(3\cdot 0)\,3e^0 = 9+3 = 12.$$

$$V(Y) = E(Y^2) - E(Y)^2 = 12 - 3^2 = \boxed{3}$$

(Y is Poisson with $\lambda = 3$)

7. The geometric distribution arises from repeating a trial with success probability p until the first success occurs. Let Y be a discrete random variable with geometric distribution with success probability p.

(4) **(a)** Give the mathematical expression for "memoryless-ness" of Y. (Partial credit for describing "memoryless" in words only.)

(6) **(b)** Compute the exact value of p for which $P(Y = 3)$ is maximized.

(a) $P(Y > a+b \mid Y > a) = P(Y > b)$. (where a, b nonnegative integers)

(b) $P(Y=3) = p(y) = p(1-p)^2$

$$\frac{d}{dp} p(1-p)^2 = \frac{d}{dp}(p^3 - 2p^2 + p) = 3p^2 - 4p + 1.$$

solving $3p^2 - 4p + 1 = 0$,

$(3p-1)(p-1) = 0$

or $p = \frac{1}{3}, 1$.

if $p = 1$, the trial never gets to the 3rd trial.

So $\boxed{p = \frac{1}{3}}$

8. Joe plays a carnival game which involves throwing a ball to knock down a pin. Joe gets as many throws as he wants, but each throw costs \$1. Joe wins a prize worth \$10 the 3rd time he knocks down a pin. Assume Joe can knock down a pin independently with probability $p = 1/4$ for each throw.

(a) What is the probability that Joe wins on the 5th throw?
(b) What is the expected number of throws required for Joe to win?
(c) If Joe wants to collect a large number of prizes, should he play the game or buy them at the store for \$10 each? (Give mathematical justification for your answer.)

(a) Let Y = the number of the toss on which the 3rd success occurs. Y is negative binomial with $p = \frac{1}{4}$, $r = 3$.

$$P(Y=5) = p(5) = \binom{5-1}{3-1}\left(\frac{1}{4}\right)^3\left(\frac{3}{4}\right)^2 = 6\left(\frac{1}{4}\right)^3\left(\frac{3}{4}\right)^2 = \cancel{\frac{27}{1352}}\ \boxed{\frac{27}{512}}$$

(b) $E(Y) = \frac{r}{p} = \frac{3}{1/4} = \boxed{12}$

(c) cost to win $= Y$, so expected cost $= 12 > 10$.

Joe should $\boxed{\text{buy them at the store}}$

9. At a bottling plant, over a long period of time it is observed that bottles are filled with soda with a mean of 11 ounces and standard deviation of .05 ounces. Find an interval of ounces in which the volume of the liquid in a randomly selected bottle falls with probability at least 8/9.

$$P(|Y-\mu| \geq k\sigma) \leq \frac{1}{k^2} \Leftrightarrow P(|Y-\mu| < k\sigma) \geq 1-\frac{1}{k^2} = \frac{8}{9} \Rightarrow \frac{1}{k^2}=\frac{1}{9}, \boxed{k=3}$$

$$P(|Y-11| < 3\cdot(.05)) \geq \frac{8}{9}$$

so $\boxed{P(11-.15 < Y < 11+.15) \geq \frac{8}{9}}$

(strict inequality)

10. The continuous random variable Y has probability density function (pdf)

$$f(y) = \begin{cases} c & \text{if } -1 \leq y \leq 0, \\ \frac{3}{20}(3+2y-y^2) & \text{if } 0 \leq y \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find c which makes $f(y)$ a pdf.
(b) Find $F(y)$, the cumulative distribution function (cdf) for Y.
(c) Compute $P\left(-\frac{1}{2} \leq Y \leq \frac{1}{2}\right)$.

(a) need $\int_{-\infty}^{\infty} f(y)\,dy = 1$.

$$\int_{-1}^{0} c\,dy + \int_0^1 \frac{3}{20}(3+2y-y^2)\,dy = 1$$

$$c + \frac{3}{20}\left(3y+y^2-\frac{y^3}{3}\right)\Big|_0^1 = 1$$

$$c + \frac{3}{20}\left(3+1-\frac{1}{3}\right) = 1$$

$$c + \frac{3}{20}\left(\frac{11}{3}\right) = 1$$

$$c = 1 - \frac{11}{20} = \boxed{\frac{9}{20}}$$

(b) $\boxed{\text{For } y < -1, \; F(y) = 0.}$

$\boxed{\text{For } -1 \leq y \leq 0,}$ $F(y) = \int_{-1}^{y} c\,dt = \boxed{(y+1)c}$

$\boxed{\text{For } 0 \leq y \leq 1,}$ $F(y) = c + \int_0^y \frac{3}{20}(3+2t-t^2)\,dt$

$$= c + \frac{3}{20}\left(3t+t^2-\frac{t^3}{3}\right)\Big|_0^y$$

$$= \boxed{c + \frac{3}{20}\left(3y+y^2-\frac{y^3}{3}\right)}$$

$\boxed{\text{For } y \geq 1, \quad F(y) = 1}$

or

$$F(y) = \begin{cases} 0, & y < -1 \\ (y+1)\frac{9}{20}, & -1 \leq y \leq 0 \\ \frac{9}{20} + \frac{3}{20}(3y+y^2-y^3/3), & 0 \leq y \leq 1 \\ 1, & y > 1 \end{cases}$$

(c) $P(-\frac{1}{2} \leq Y \leq \frac{1}{2})$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} f(y)\,dy$$

$$= F\left(\tfrac{1}{2}\right) - F\left(-\tfrac{1}{2}\right)$$

$$= \left[c + \frac{3}{20}\left(\frac{3}{2}+\frac{1}{4}-\frac{1}{3\cdot 8}\right)\right] - \frac{c}{2} = \frac{c}{2} + \frac{41}{160} = \boxed{77/160}$$