SHOW WORK FOR FULL CREDIT NO CALCULATORS

1. A deck of cards consists of 52 cards, represented by the product set $\{2, \ldots, 10, J, Q, K, A\} \times \{C, D, H, S\}$. That is, each card has one of 13 *face values* and one of 4 *suits*. The deck is shuffled to be completely random, and cards are dealt so that (i) Player 1 receives 4 cards, (ii) Player 2 receives 4 cards, and (iii) 2 cards are placed on the table. The remaining cards are left unused.
(a) What is the probability that Player 1 is dealt 4 cards with the same face value?
(b) Given that the two cards on the table are $(2, C)$ and $(4, H)$, what is the probability that Player 1 has been dealt 4 cards with the same face value?
(c) What is the probability that Player 1 is dealt 4 cards with the same suit?

2. Medical case histories indicate that different illnesses may produce identical symptoms. Suppose that a particular set of symptoms, denoted H, occurs only when any one of three illnesses I_1, I_2, or I_3, occurs. Assume that the simultaneous occurrence of more than one of these illnesses is impossible and that

$$P(I_1) = .05, \qquad P(I_2) = .001, \qquad \text{and} \quad P(I_3) = .02.$$

The probabilities of developing the set of symptoms H, given each of the illnesses, are known to be

$$P(H|I_1) = .90, \qquad P(H|I_2) = .95, \qquad \text{and} \quad P(H|I_3) = .75.$$

(a) If a person is selected at random, what is the probability that the person exhibits the symptoms H?
(b) Select a person at random. Given that the person exhibits the symptoms H, what is the probability that the person has illness I_3?

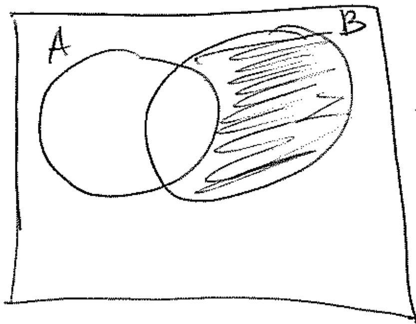
(a) $P(H) = P(H|I_1)P(I_1) + P(H|I_2)P(I_2) + P(H|I_3)P(I_3) + P(H|\overline{I_1 \cup I_2 \cup I_3}) \cdot P(\overline{I_1 \cup I_2 \cup I_3})$

$= (.90)(.05) + (.95)(.001) + (.75)(.02) + 0(.071)$

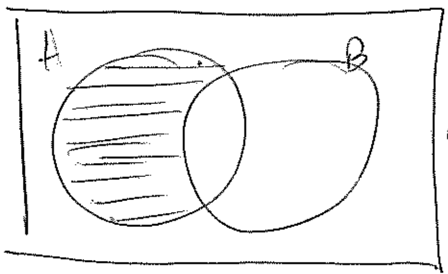
$= \boxed{.045 + .00095 + .015}$

(b) $P(I_3|H) = \dfrac{P(H|I_3) \cdot P(I_3)}{P(H)} = \boxed{\dfrac{.015}{.045 + .00095 + .015}}$

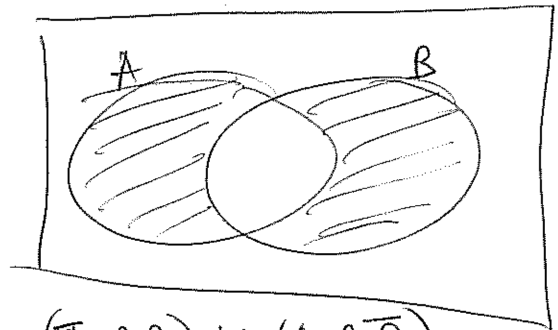
3. Illustrate the following sets on three *separate* Venn diagrams: $\overline{A} \cap B$, $A \cap \overline{B}$, and $(\overline{A} \cap B) \cup (A \cap \overline{B})$



$\overline{A} \cap B$

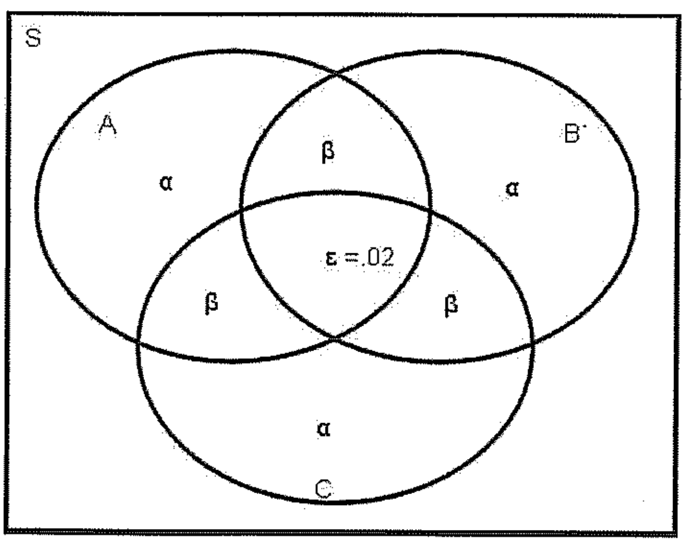


$A \cap \overline{B}$



$(\overline{A} \cap B) \cup (A \cap \overline{B})$

4. Events $A_1, \ldots, A_n$ are *k-wise independent* provided $\mathrm{P}(A_{i_1} \cap \cdots \cap A_{i_k}) = \mathrm{P}(A_{i_1}) \cdots \mathrm{P}(A_{i_k})$ for any choice of k distinct events from the original n events.
(a) Design a probability space by finding α and β in the appropriate sections of the Venn diagram so that A, B, and C are 2-wise independent but not 3-wise independent. (For example, $\mathrm{P}(A \cap B \cap \overline{C}) = \beta$.) You are given that $\mathrm{P}(A \cap B \cap C) = \epsilon = .02$.
(b) Compute $\mathrm{P}(A)$, $\mathrm{P}(A \cap B)$, and $\mathrm{P}(A \cap B \cap C)$ given your choices of α and β in **(a)**.



(a) $\boxed{\beta = .07}$

$\alpha = .3 - .07 - .07 - .02$

$\boxed{\alpha = .14}$

(b) $P(A) = \alpha + 2\beta + \epsilon$

$\boxed{P(A) = .3}$

$P(A \cap B) = \beta + \epsilon = .07 + .02$

$\boxed{P(A \cap B) = .09}$

$\boxed{P(A \cap B \cap C) = \epsilon = .02}$

5. An experiment consists of flipping a fair coin and rolling a 6-sided die with the following properties: (i) each even die outcome is equally likely, (ii) each odd die outcome is equally likely, and (iii) an odd die outcome is exactly twice as likely as an even die outcome. The outcome of the coin is independent of the outcome of the die.
   (a) List all of the sample points (simple events) of the experiment along with their probabilities.
   (b) What is the probability that the the coin is tails and the die roll is ≤ 3?
   (c) Given that the die roll is odd, what is the probability that the coin toss is heads?

(a) $P(\text{die}=1) = 2 \cdot P(\text{die}=2)$

$x = 2y$

$3x + 3y = 1$

system solved by

$x = \frac{2}{9} \quad y = \frac{1}{9}$.

(H,1)	$\frac{2}{18}$	H,5	$\frac{2}{18}$	T,4	$\frac{1}{18}$
(H,2)	$\frac{1}{18}$	H,6	$\frac{1}{18}$	T,5	$\frac{2}{18}$
(H,3)	$\frac{2}{18}$	T,1	$\frac{2}{18}$	T,6	$\frac{1}{18}$
(H,4)	$\frac{1}{18}$	T,2	$\frac{1}{18}$		
		T,3	$\frac{2}{18}$		

(b) $P(\text{coin tails and die} \leq 3) = \frac{2}{18} + \frac{1}{18} + \frac{2}{18} = \boxed{\frac{5}{18}}$

(c) $P(\text{heads} \mid \text{die odd}) = \frac{P(\text{heads} + \text{die odd})}{P(\text{die odd})}$

$= \frac{\frac{2}{18} + \frac{2}{18} + \frac{2}{18}}{\frac{2}{9} + \frac{2}{9} + \frac{2}{9}}$

(events are independent) $= \boxed{\frac{1}{2}} = P(\text{heads})$

6. A coin is flipped twice in a row. The probability that the first toss is heads is 1/2. The probability that the second toss is heads is also 1/2. However, The probability that both tosses are heads is 1/8, and the probability that both tosses are tails is 1/8 (perhaps due to some mysterious quantum coupling).
   (a) Label each of the four events to which probabilities have been assigned in the problem statement.
   (b) Write each sample point of the experiment in terms of the events in **(a)**.
   (c) Determine the probability that the first coin flip is tails and the second coin flip is heads.

(a) A_1 = event first coin is heads

A_2 = event second coin is heads

E_{HH} = (simple) event both coins are heads.

E_{TT} = (simple) event both coins are tails.

(b) $S = \{HH, HT, TH, TT\}$

$\{HH\} = A_1 \cap A_2 = E_{HH}$

$\{HT\} = A_1 \cap \bar{A}_2$

$\{TH\} = \bar{A}_1 \cap A_2$

$\{TT\} = \bar{A}_1 \cap \bar{A}_2 = E_{TT}$

(c) $P(TH) = P(\bar{A}_1 \cap A_2)$

we know $P(A_2) = P(A_1 \cap A_2) + P(\bar{A}_1 \cap A_2)$

$P(*H) = P(HH) + P(TH)$

$\frac{1}{2} = \frac{1}{8} + P(TH)$

so $P(TH) = \boxed{\frac{3}{8}}$

7. Bowl 1 contains 10 plain cookies and 30 chocolate chip cookies. Bowl 2 contains 20 plain cookies and 20 chocolate chip cookies. An experiment is conducted as follows: (i) a bowl is selected at random, with either being equally probable, and (ii) a cookie from within the chosen bowl is selected at random with each being equally probable.
   (a) What is the probability that a plain cookie is selected given that Bowl 1 is selected?
   (b) What is the probability that a plain cookie is selected?
   (c) What is the probability that Bowl 1 was selected given that a plain cookie has been selected?

	Plain	choc chip
Bowl 1	10	30
Bowl 2	20	20

(a) $P\left(\text{plain selected} \mid \text{Bowl 1 selected}\right) = \boxed{\frac{10}{40}}$

(b) $P(\text{plain selected}) = P(\text{plain} \mid \text{bowl 1})P(\text{bowl 1}) + P(\text{plain} \mid \text{bowl 2})P(\text{bowl 2})$

$= \frac{10}{40} \cdot \frac{1}{2} + \frac{20}{40} \cdot \frac{1}{2} = \frac{30}{80} = \boxed{\frac{3}{8}}$

(c) $P(\text{Bowl 1} \mid \text{plain})$

$$= \frac{P(\text{plain} \mid \text{Bowl 1})P(\text{Bowl 1})}{P(\text{plain} \mid \text{Bowl 1})P(\text{Bowl 1}) + P(\text{plain} \mid \text{Bowl 2})P(\text{Bowl 2})}$$

$$= \frac{\left(\frac{10}{40}\right)\left(\frac{1}{2}\right)}{\left(\frac{10}{40}\right)\left(\frac{1}{2}\right) + \left(\frac{20}{40}\right)\left(\frac{1}{2}\right)} = \frac{\frac{10}{80}}{\frac{30}{80}} = \boxed{\frac{10}{30}}$$

8. A package courier has 4 packages which each have a distinct destination. Due to a malfunction at the office, the destinations are scrambled, so that each package still has exactly one destination, but each way of assigning packages to destinations is equally likely. What is the probability that exactly 2 packages will reach their correct destinations?

probability $= \frac{\binom{4}{2} \cdot 1 \cdot 1}{4!} = \boxed{\frac{1}{4}}$

Packages: 1, 2, 3, 4

destinations: ordered blanks _ _ _ _.

The question requires the number of permutations in which exactly two elements are in their rightful locations.

Perform in sequence

- select the two packages to go to the right place. $\binom{4}{2}$ ways.
- put them in their right place. 1 way.
- There are two numbers left, and two blanks.

  A, B   one placement is in order,
  _ _   one is out of order (not matching destination)
- only one way to place other 2 packages.