PRINT Last name:____________________ First name:____________________

Signature:____________________ Student ID:____________________

Math 454 Exam 2, Fall 2005

I. Examples and Counterexamples. (7 pts ea.) Do not give proofs, but clearly indicate your proposed example or counterexample.

1. Give an example of a simple graph G such that $G - e$ for any edge $e \in E(G)$ is a spanning tree of G.

2. Pick some $n \geq 8$. Give an example of a tree T on vertices $\{1, \ldots, n\}$ such that every number that appears in the Prüfer code of T appears exactly twice.

3. Give an example of a graph for which the maximum matching has a different size than the minimum vertex cover (i.e., $\alpha'(G) \neq \beta(G)$). Indicate a maximum matching and a minimum vertex cover on your graph.

4. Find an augmenting path in the following graph, whose matching edges are **red** and whose non-matching edges are black. After increasing the size of the matching via that augmenting path, is there another augmenting path (Circle: YES / NO)?

5. Give an example of a simple graph for which $\kappa(G) < \delta(G)$ (vertex connectivity is less than minimum degree).

6. Give an example of a simple graph in which there is a minimum edge cut $[S, \overline{S}]$ for which $|S| \geq 2$.

II. Constructions and Algorithms. (14 pts ea.) Do not write proofs, but do give clear, concise answers.

7. Use any of the methods we studied to determine the number of spanning trees of the following graph. Briefly explain any fact that you use, but you don't have to prove it.

8. Complete the Hungarian algorithm on the following edge weight matrix of a graph in order to find a maximum weight matching. Show the minimum weight cover which verifies optimality. (The graph G is a simple X,Y-bigraph with $|X| = |Y| = 5$. The entry in the ith row and jth column of the matrix W is the weight of edge $x_i y_j$.)

$$W = \begin{array}{|ccccc|} \hline 10 & 9 & 6 & 4 & 3 \\ 7 & 8 & 9 & 7 & 10 \\ 6 & 9 & 2 & 5 & 8 \\ 8 & 8 & 1 & 4 & 7 \\ 10 & 9 & 7 & 5 & 6 \\ \hline \end{array} \qquad [u_i + v_j - w_{i,j}] : \quad \begin{array}{c|ccccc|} & 2 & 2 & 0 & 0 & 1 \\ \hline 8 & 0 & 1 & 2 & 4 & 6 \\ 9 & 4 & 3 & 0 & 2 & 0 \\ 7 & 3 & 0 & 5 & 2 & 0 \\ 6 & 0 & 0 & 5 & 2 & 0 \\ 8 & 0 & 1 & 1 & 3 & 3 \\ \hline \end{array}$$

III. Proofs. (15 pts ea.) Part of the score is determined by careful formatting of the proof (forward and reverse directions, assumptions, conclusions, etc.). Partial credit will be awarded for this as well.

9. Let G be a graph with positive edge weights $w(e)$ for any $e \in E(G)$.
   Let $F \subseteq E(G)$ be the set of edges with smallest weight c.
   Now prove that F is contained in every minimum spanning tree T of G if and only if F is acyclic. (Refer to Figure 8ab for an illustration, but be sure to write the proof for the general case.)

10. Prove that a tree T has a perfect matching if and only if $o(T - v) = 1$ for every $v \in V(T)$. (Hints. Recall that $o(T - v)$ is the number of odd components of $T - v$. For one of the directions, to which vertex must v be matched?)