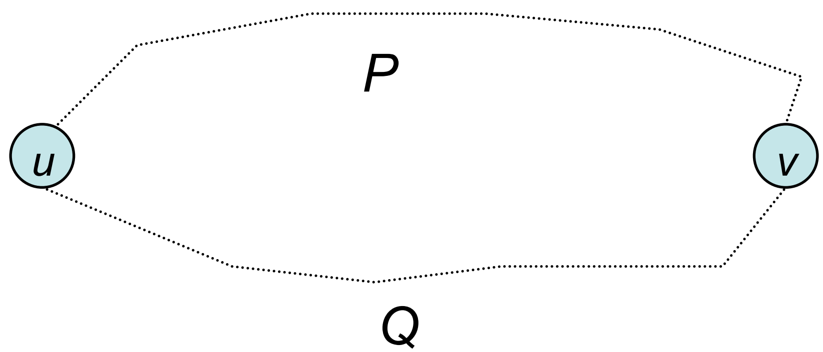
4.2 *k*-connected graphs

4.2 A characterization for 2-connectedness

<u>Thm 4.2.2 (Whitney)</u> A graph *G* with ¸3 vertices is 2-connected iff 8 *u*,*v*2 V(*G*) there exist ¸2 internally disjoint *u*,*v*-paths in *G*.

((Easy) Let *S*={*w*}µV(*G*). Let *u*,*v*2 *G*-*S*.

Let *P*,*Q* be internally disjoint paths in *G*



w can be on at most one of these paths, so removing *w* fails to disconnect *u* and *v*.

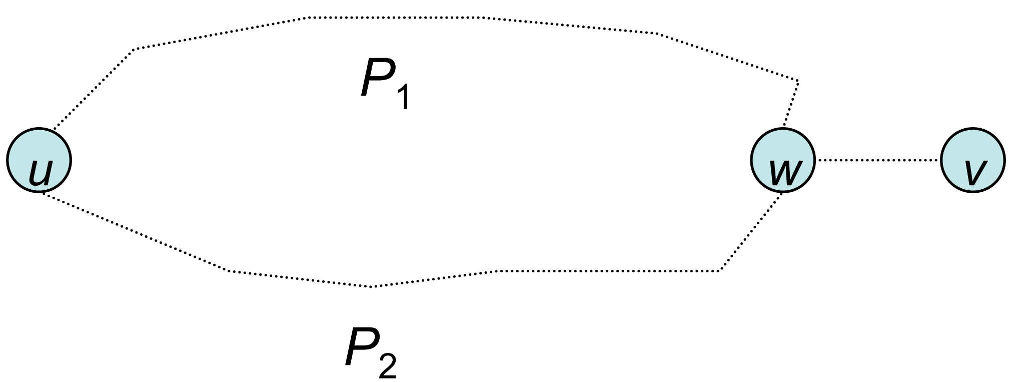
4.2 A characterization for 2-connectedness

<u>Thm 4.2.2 (Whitney)</u> A graph *G* with ¸3 vertices is 2-connected iff 8 *u*,*v*2 V(*G*) there exist ¸2 internally disjoint *u*,*v*-paths in *G*.

()) Assume *G* is 2-connected. Let *u*,*v*2V(*G*).

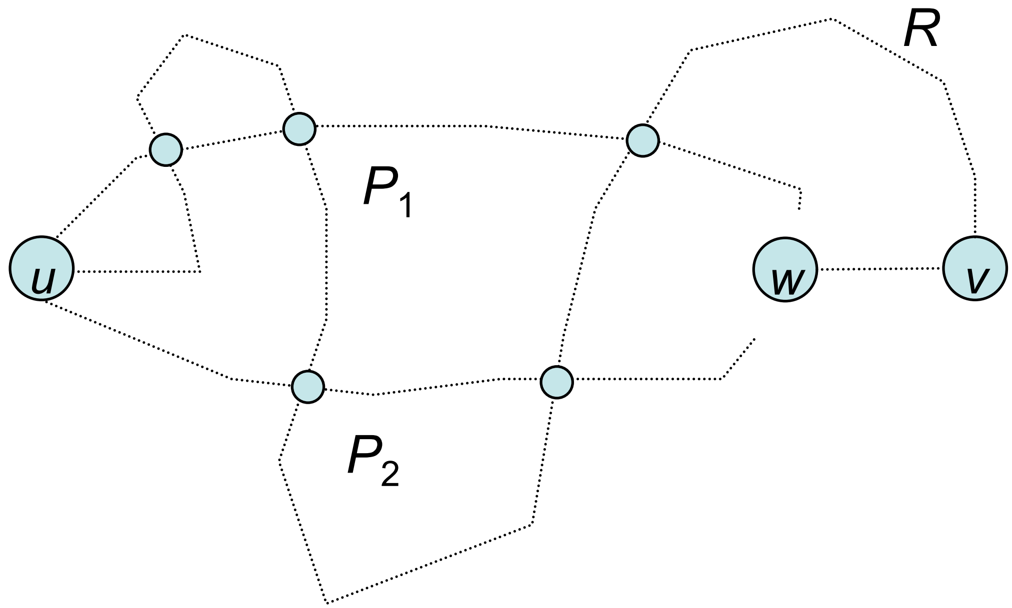
Induction on d(*u*,*v*):

w is closer to *u*, and so there exist *u*,*w*-paths



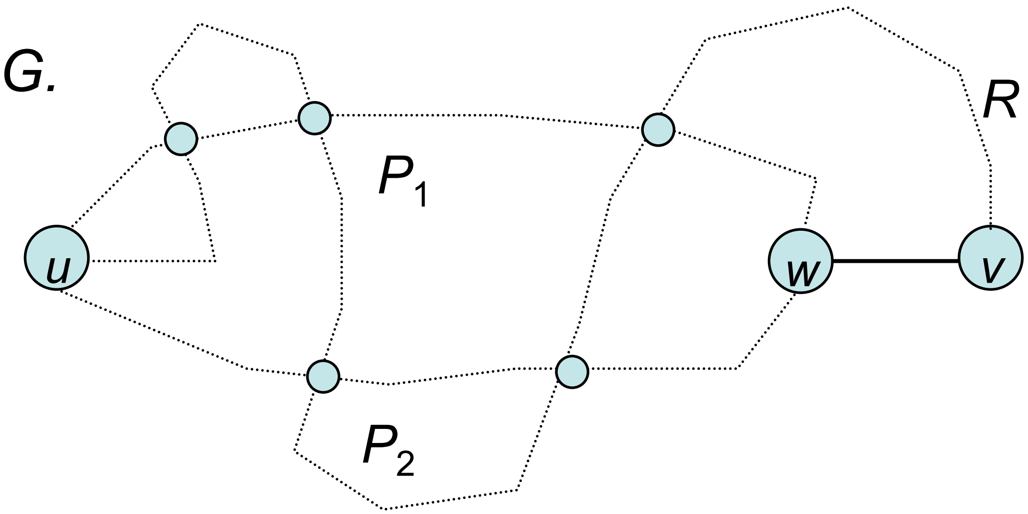
4.2 A characterization for 2-connectedness

G-w is connected: there is a *u*,*v* path in *G-w*, called *R*.

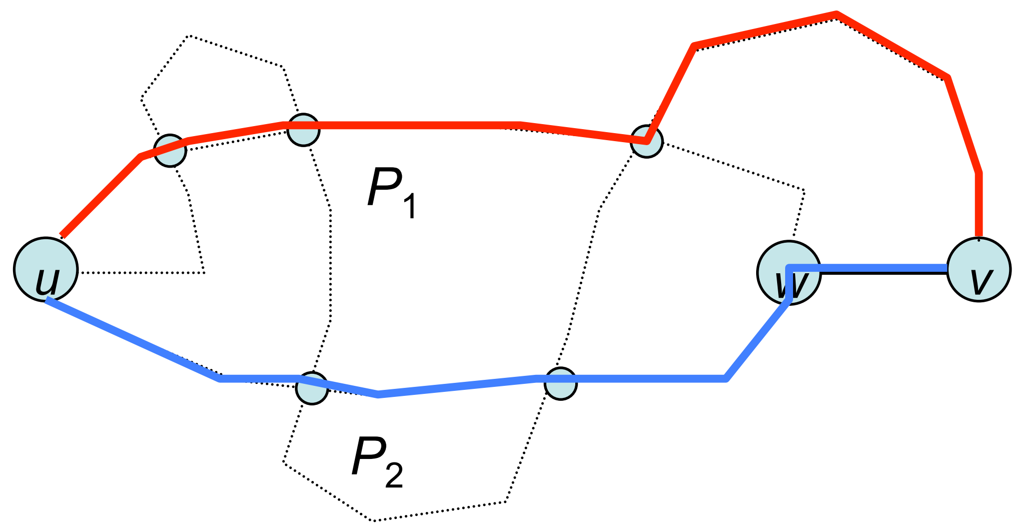


4.2 A characterization for 2-connectedness

Now look at the original *G*.



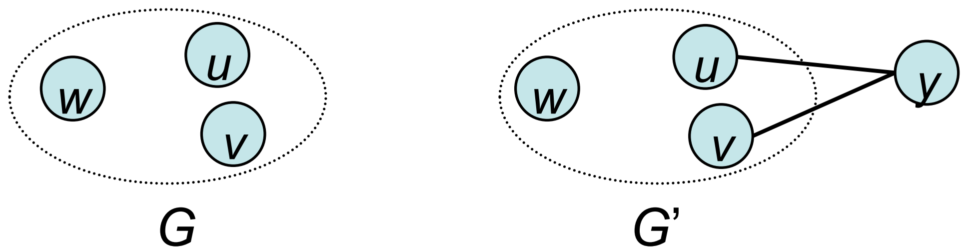
Find two internally disjoint *u*,*v*-paths



4.2 Expansion Lemma

Lemma 4.2.3 (Expansion Lemma). If G is a k-connected (loopless) graph, and G' is obtained by adding a new vertex y with ¸k neighbors in G, then G' is k-connected.

Proof for k=2.



n(G)¸3 is required by k=2.

No single vertex can be cut to disconnect G'.

Try u: G-u still connected, and y connected through v,

so G' is still connected.

Try y: G'-y=G is connected.

Try w: Like first case, but y connected to both u and v in G'-w

4.2 Extended characterization for 2-connectedness

Thm 4.2.4 Let a graph *G* have ¸3 vertices. The following are equivalent (TFAE):

(A) G is connected and has no cut-vertex,

(B) For all *x*,*y*2 V(*G*), there exist internally disjoint *x*,*y*-paths,

(C) For all *x*,*y*2 V(*G*), there is a cycle through *x* and *y,*

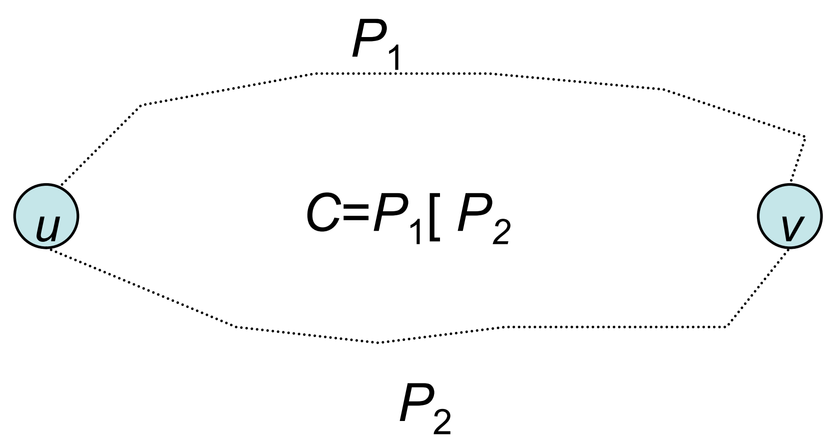
(D) δ(*G*)¸1, and every pair of edges lies on a common cycle.

Proof.

(A) , (B) already done.

(B) , (C)

cycle iff two disjoint paths



4.2 Extended characterization for 2-connectedness

(D))(C) (δ(*G*)¸1, any 2 edges are in some same cycle)any 2 vertices are in some same cycle)

Let *x*,*y*2 V(*G*). Min degree forces each incident to an edge:

<u>Case 1</u>. x$y

n(*G*)¸3, so there is a third vertex *z*. A cycle through these two edges also goes through *x*,*y*.

Case 2. x$/y

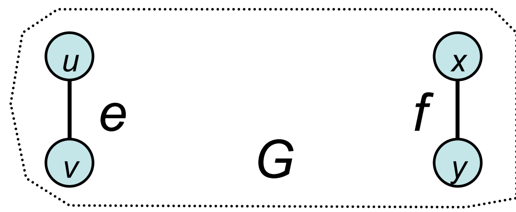
A cycle through these two edges also goes through *x*,*y*.

4.2 Extended characterization for 2-connectedness

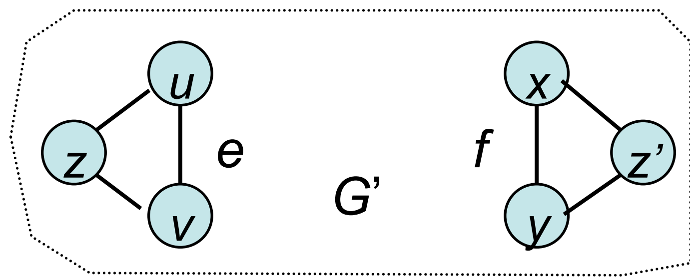
(A)Æ(C))(D): n(*G*)¸3 and *G* is connected) δ(*G*)¸1.

Let *e*,*f*2E(*G*) be edges,

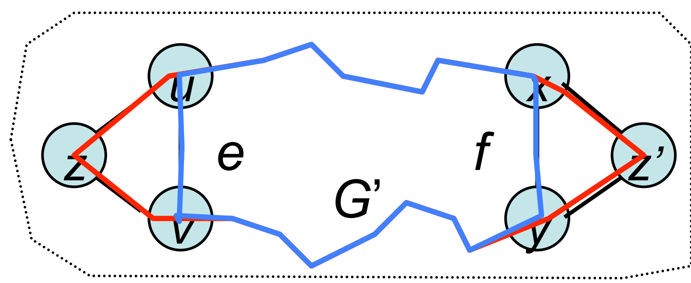
with *e*=*uv*, *f*=*xy*.



Construct *G*' by two expansions. *G*' is still 2-connected

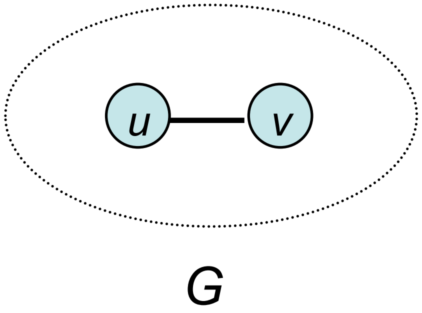


(C))*G*' has a cycle through *x*,*y*. Edit the cycle to get a cycle Through *e*,*f*

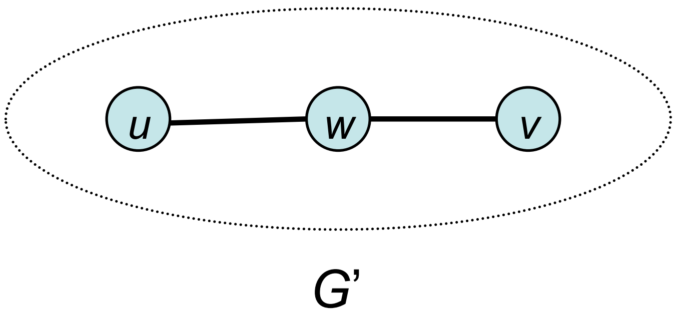


Definition of subdivision

G with edge *uv*



G' from *G* by subdividing edge *uv*



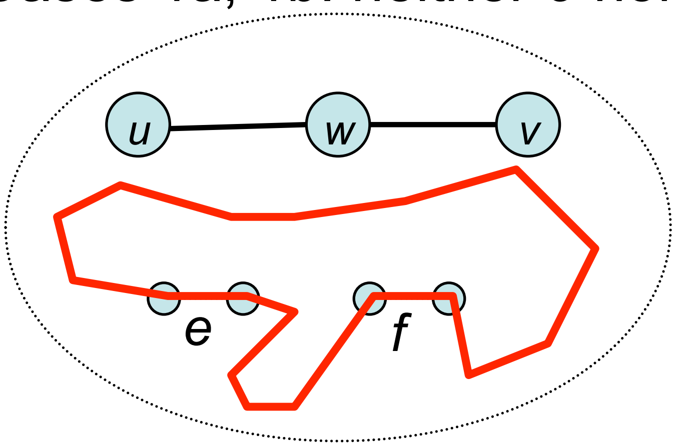
Subdivisions preserve 2-connectedness

<u>Corollary 4.2.6</u>. If *G* is 2-connected, then so is the graph *G*' obtained from *G* by subdividing an edge of *G*.

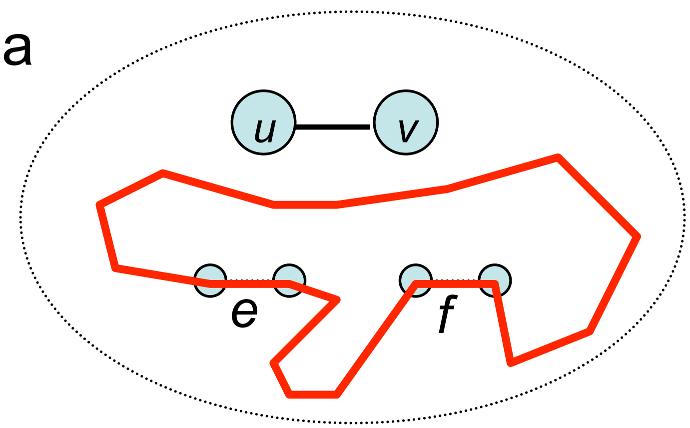
Proof. *G* 2-connected)n(*G*)¸3. Let *e*,*f*2E(*G*'). We want a common cycle containing *e*,*f*.

Cases 1a, 1b: neither *e* nor *f* within subdivision

Case 1a



G'



G

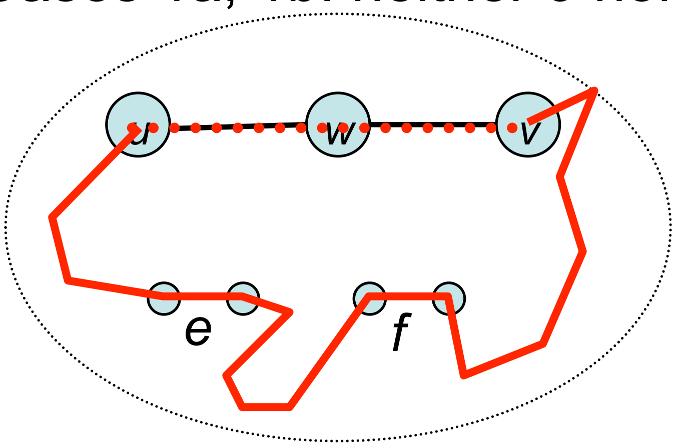
Subdivisions preserve 2-connectedness

<u>Corollary 4.2.6</u>. If *G* is 2-connected, then so is the graph *G*' obtained from *G* by subdividing an edge of *G*.

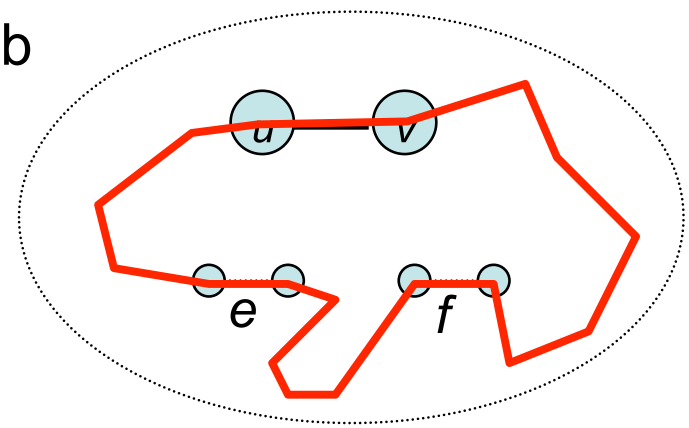
Proof. *G* 2-connected)n(*G*)¸3. Let *e*,*f*2E(*G*'). We want a common cycle containing *e*,*f*.

Cases 1a, 1b: neither *e* nor *f* within subdivision

Case 1b



G'



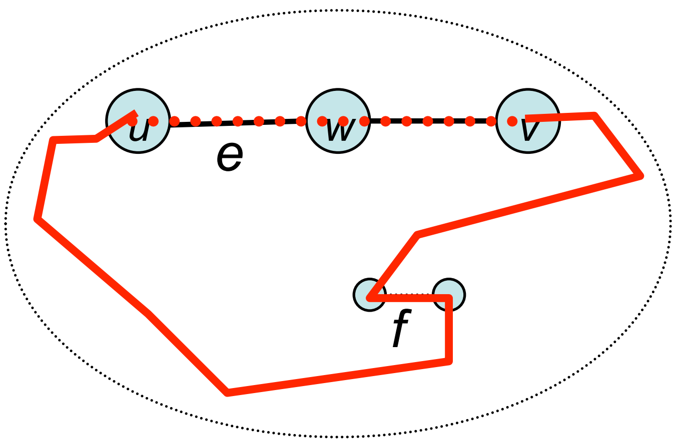
G

Subdivisions preserve 2-connectedness

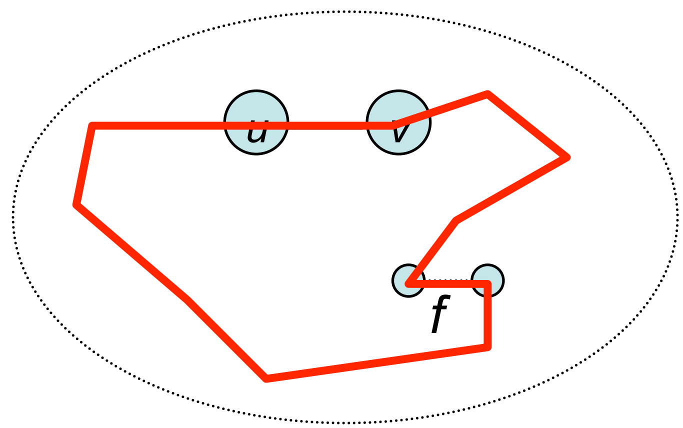
Corollary 4.2.6. If *G* is 2-connected, then so is the graph *G*' obtained from *G* by subdividing an edge of *G*.

Proof. *G* 2-connected)n(*G*)¸3. Let *e*,*f*2E(*G*'). We want a common cycle containing *e*,*f*.

Cases 2: *e* is within subdivision, but not *f*



G'



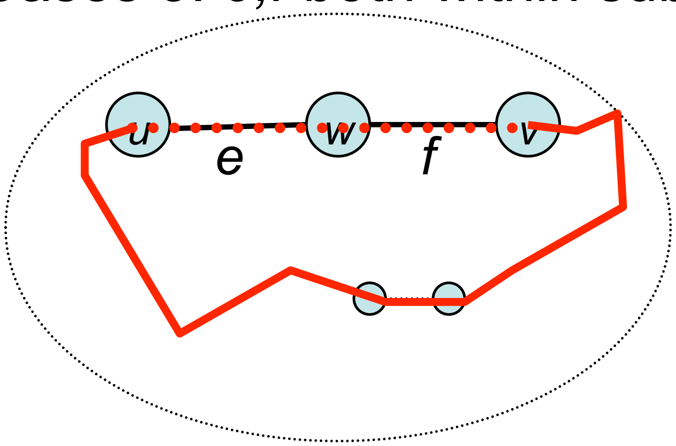
G

Subdivisions preserve 2-connectedness

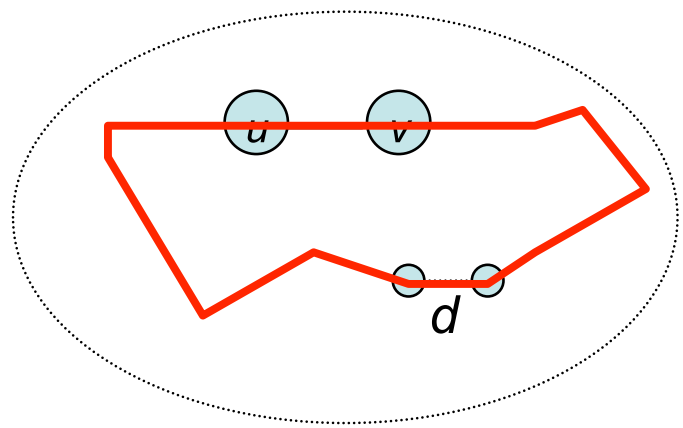
<u>Corollary 4.2.6</u>. If *G* is 2-connected, then so is the graph *G*’ obtained from *G* by subdividing an edge of *G*.

Proof. *G* 2-connected)n(*G*)¸3. Let *e*,*f*2E(*G*’). We want a common cycle containing *e*,*f*.

Cases 3: *e,f* both within subdivision. Edge *d* must exist.



G’



G

Ear decompositions and 2-connected graphs

<u>Definition</u> An ear decomposition of G is a sequence of graphs

$$P_0,P_1,\ldots,P_i,\ldots,P_k$$

of subgraphs of G such that:

(1) P_0 is a cycle in G,

(2) For all i¸1, P_i is an **ear** of $P_0,P_1,\ldots,P_{i-1}$, meaning

(i) P_i is a path

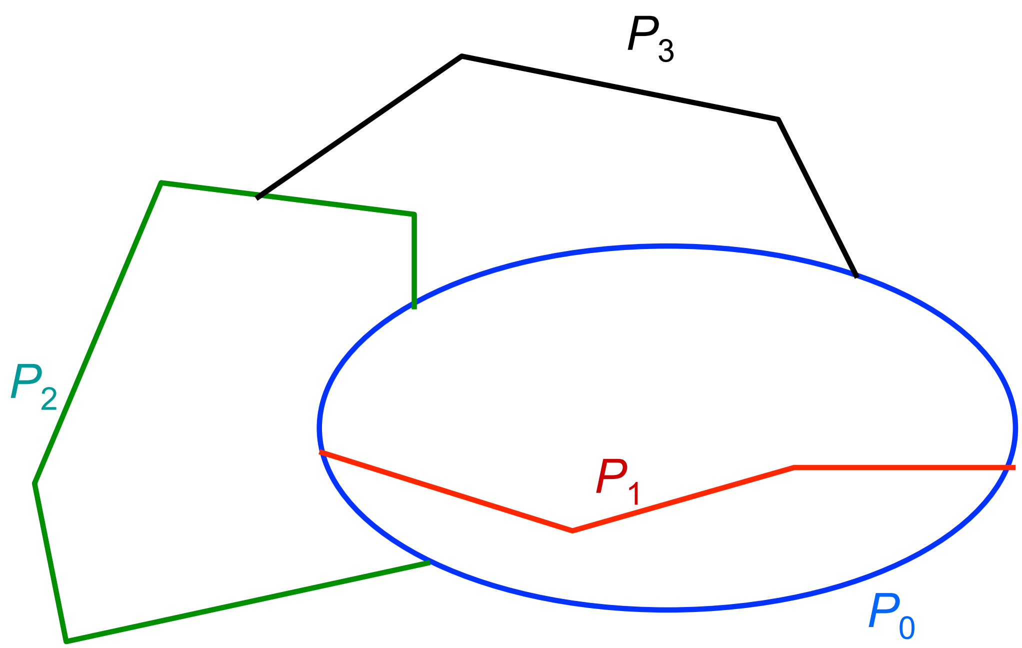
(ii) P_i is contained in a cycle of $P_0,P_1,\ldots,P_{i-1},P_i$

(iii) P_i is maximal w.r.t. internal vertices having degree 2 in $P_0,P_1,\ldots,P_{i-1},P_i$

(2) $P_0,P_1,\ldots,P_i,\ldots,P_k$ decompose G

Ear decompositions and 2-connected graphs

Example Ear decomposition of a graph.

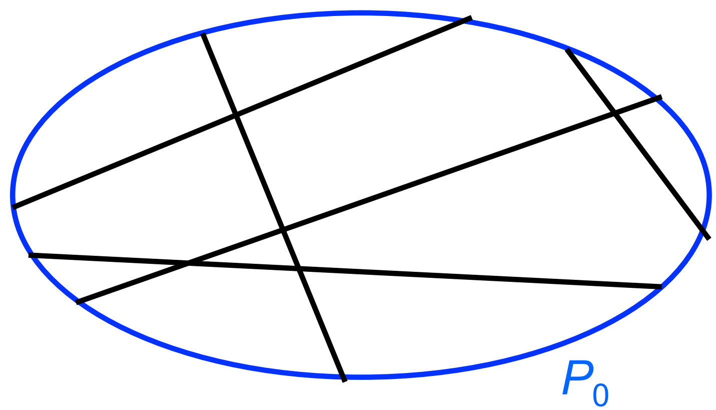


Note that P_3 cannot come before P_2 in the order.

Ear decompositions and 2-connected graphs

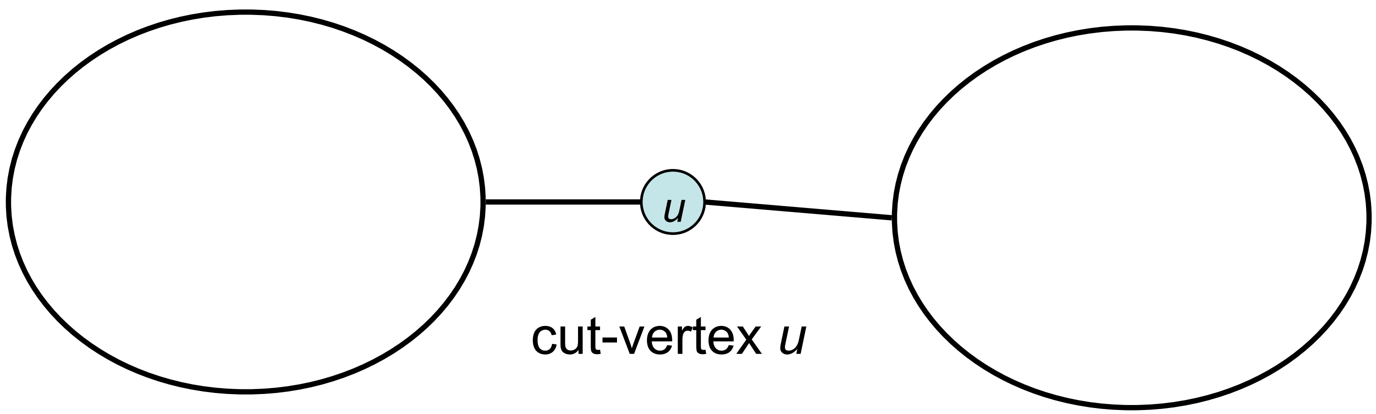
Example Highly connected graphs have ear decompositions.

Suppose G has a cycle containing all of its vertices. Then all other edges can be added as ears one at a time, arbitrarily.



Ear decompositions and 2-connected graphs

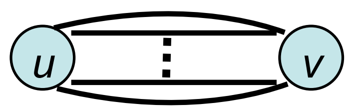
Example Graphs with connectivity 1 do not have ear decompositions.



The ear decomposition must start on one side of u since there is no cycle containing u. But then there is no way to add u within some path P_i that is an ear!

Ear decompositions and 2-connected graphs

<u>…except</u> The 2-vertex graph with 2 or more parallel edges. (Remember, in Chapter 4 there are no loops!)



Set P_0 equal to a 2-cycle, P_1,P_2,… handle any remaining edges.

Ear decomposition iff 2-connected (and n(*G*)¸3)

<u>Theorem 4.2.8 (Whitney)</u> *G* has an ear decomposition iff *G* is 2-connected. We add the condition n(*G*)¸3 to both sides to handle the exceptional case.

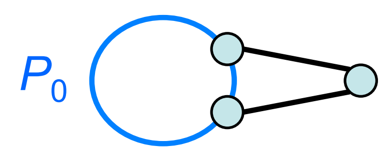
Proof. (()

Let $P_0,P_1,\ldots,P_k$ be an ear decomposition of *G*.

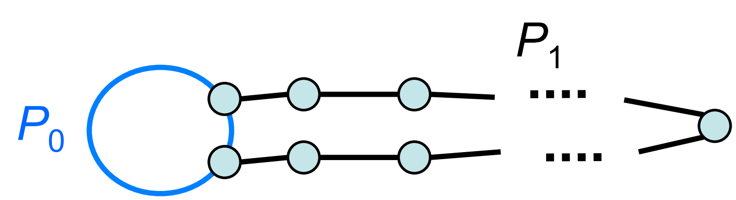
Assume P_0 is larger than a 2-cycle (otherwise… exercise).

Cycles are 2-connected.

Get P_0[P_1 by expansion…



And then repeated subdivision.



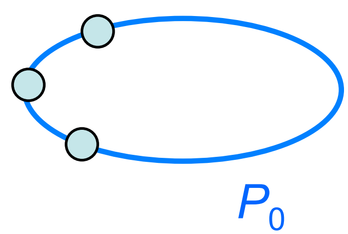
2-connectivity is preserved for both operations. Repeat for $P_2,\ldots$

Ear decomposition iff 2-connected (and n(*G*)¸3)

<u>Theorem 4.2.8 (Whitney)</u> Proof. ())

Assume *G* is 2-connected. By Thm. 4.2.4, *G* has two edges not both with the same neighbors. And *G* has a cycle containing these two edges. This cycle must have at least one other edge.

Either way, a cycle with ¸3 edges exists in G. Call it P_0.

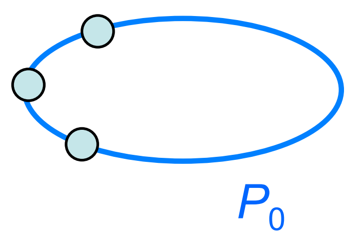


Ear decomposition iff 2-connected (and n(*G*)¸3)

Theorem 4.2.8 (Whitney) Proof. ())

Assume *G* is 2-connected. By Thm. 4.2.4, *G* has two edges not both with the same neighbors. And *G* has a cycle containing these two edges. This cycle must have at least one other edge.

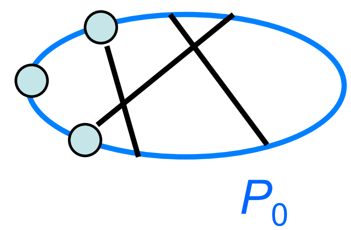
Either way, a cycle with ¸3 edges exists in G. Call it P_0.



Ear decomposition iff 2-connected (and n(G)¸3)

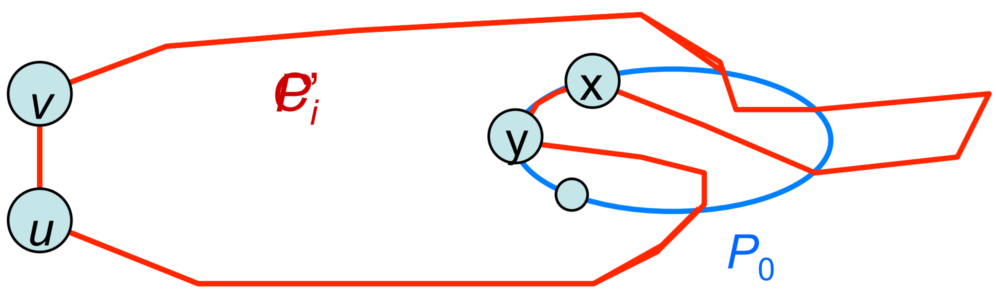
<u>Theorem 4.2.8 (Whitney)</u> Proof. () continued)

Add all edges between two vertices of the cycle; call them $P_1,\ldots,P_j$.



If there is an edge uv not in P_0, Thm. 4.2.4 says there is a cycle C' containing uv and an edge on P_0.

Delete vertices between the first and last vertices touched within P_0.



Repeat combinations of these two steps until G is decomposed.

Material skipped by slides

4.2.10 Theorem. A graph is 2-connected iff it has a closed-ear decomposition, and every cycle in a 2-edge-connected graph is the initial cycle in some such decomposition.

4.2.13 Theorem (Robbins 1939) A graph has a strong orientation iff it is 2-edge-connected.

Plus supporting definitions and examples on closed ears, closed-ear decompositions, connectivity for digraphs, (directed) ears in digraphs. See pp.164-166.

Local connectivity and Menger's Theorem

Local connectivity considers the number of alternative paths between a pair of vertices x,y, and the minimum size structure needed to be deleted to disconnect x from y.

Global connectivity considers the number of alternative paths between any pair of vertices, and the minimum size structure needed to be deleted to disconnect the graph.

We have seen that (for $n(G)$ ¸ 3) 2-connectedness is equivalent to there existing 2 internally disjoint paths between any pair of vertices. We want to extend this idea:

Locally: compare alternative x,y-paths versus minimum x,y-cuts

Globally: find vertex pairs optimizing the local values

Definitions for local connectivity

<u>4.2.15 Defn.</u> Let {x,y}2V(G), SµV(G)-{x,y}, FµE(G), X,YµV(G)

(1) S is an x,y-separator (x,y-cut) if G-S has no x,y-path

(1') $\kappa(x,y)$ = minimum size of an x,y-separator

(2) $\lambda(x,y)$ = maximum # of pairwise internally disjoint x,y-paths

(3) F is an x,y-disconnecting set if G-F has no x,y-path

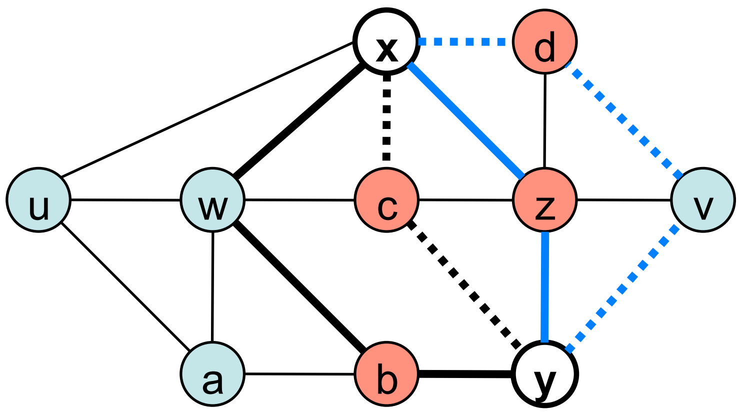
(3') $\kappa'(x,y)$ = minimum size of an x,y-disconnecting set

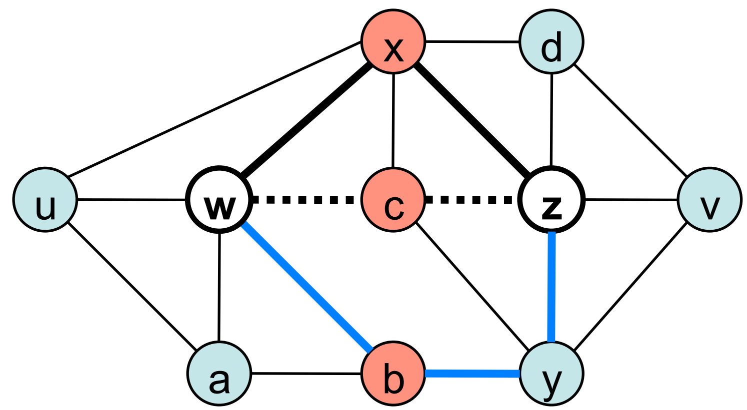
(4) $\lambda'(x,y)$ = maximum # of pairwise edge-disjoint x,y-paths

(5) An X,Y-path starts in X, ends in Y, otherwise avoids X[Y

Examples for local connectivity (1)

4.2.16 Example



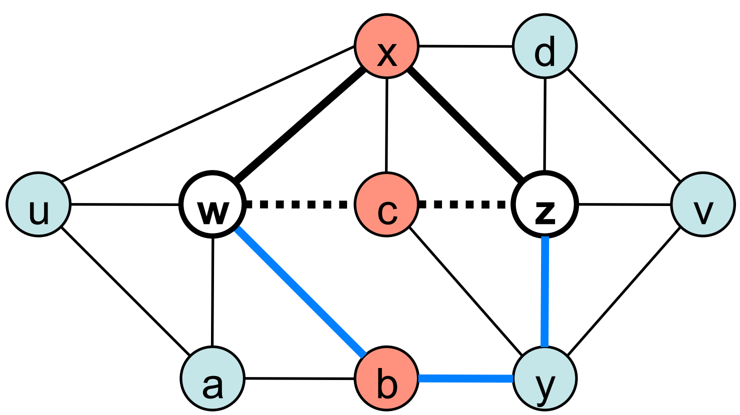


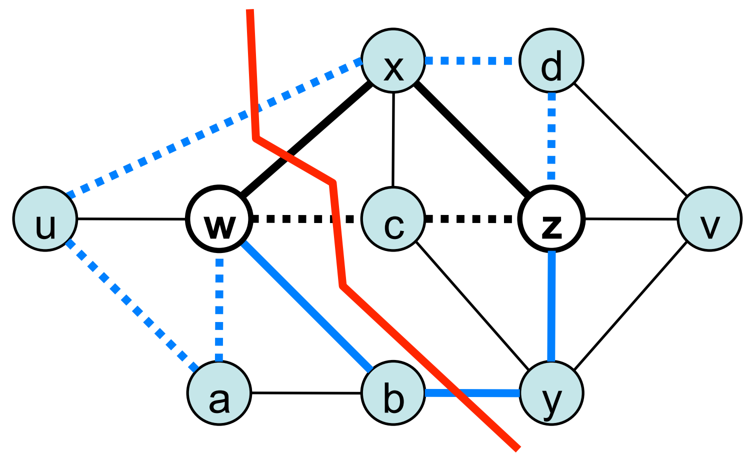
x,*y*-connectivity
$\lambda(x,y)$ ¸ 4 (4 int. disj. *x*,*y*-paths)
$\kappa(x,y)$ · 4 (cut =)

w,*z*-connectivity
$\lambda(w,z)$ ¸ 3 (3 int. disj. *w*,*z*-paths)
$\kappa(w,z)$ · 3 (cut =)

Examples for local connectivity (2)

4.2.16 Example





w,*z*-connectivity (from prev. slide)
λ(*w*,*z*) ¸ 3 (3 int. disj. *w*,*z*-paths)
κ(*w*,*z*) · 3 (cut =)

w,*z*-**edge**-connectivity
λ'(*w*,*z*) ¸ 4
(4 int. edge-disj. *w*,*z*-paths)
κ'(*w*,*z*) · 4
(disconnecting set touches)

Karl Menger

(http://www.iit.edu/csl/am/about/menger/about.shtml)

1902 born in Vienna
1920-1924 Ph.D. in Mathematics, University of Vienna
[1944-1][1944-6] 1946-1971 Professor of Mathematics, IIT
1985 died in Highland Park near Chicago

In 1932 Menger published *Kurventheorie* which contains the famous *n*-Arc Theorem:

> Let *G* be a graph with *A* and *B* two disjoint *n*-tuples of vertices. Then either *G* contains *n* pairwise disjoint *AB*-paths (each connecting a point of *A* and a point of *B*), or there exists a set of fewer than *n* vertices that separates *A* and *B*.

Other Research Areas: Theory of Curves and Dimension Theory, A General Theory of Length and the Calculus of Variations, Probabilistic Metric Spaces, New Foundations for the Bolyai-Lobachevsky Geometry, and many others

Menger's Theorem

<u>4.2.17 Theorem (Menger 1927).</u> If x,y are vertices of a graph G and $xy \notin E(G)$, then $\kappa(x,y) = \lambda(x,y)$.

<u>Proof (assume *G* simple)</u>

$\kappa(x,y) \geq \lambda(x,y)$ is easy: an x,y-cut must have ≥ 1 vertex from each of a set of pairwise internally disjoint x,y-paths.

$\kappa(x,y) \leq \lambda(x,y)$: Set $k = \kappa(x,y)$. We find k pairwise internally disjoint x,y-paths by induction on $n(G)$.

<u>Base case ($n(G)=2$)</u>

$xy \notin E(G)$ means G is empty, and $\kappa(x,y) = \lambda(x,y) = 0$.

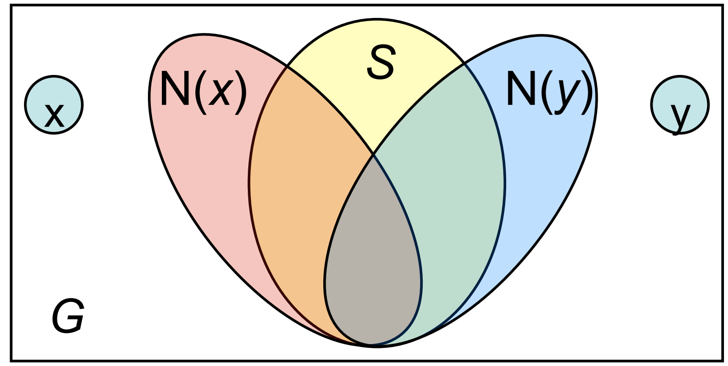
<u>Inductive step ($n(G)>2$)</u>

Let S be a minimum x,y-cut (with $|S|=k$). Consider cases…

Menger's Theorem (2)

Structure of G determines cases.

<u>Fact</u> S does not properly contain N(x) or N(y). N(x) and N(y) are themselves x,y-cuts.



<u>Case 1.</u> 9S with Sµ/N(x)[N(y).

Define V_1=vertices of all x,S-paths

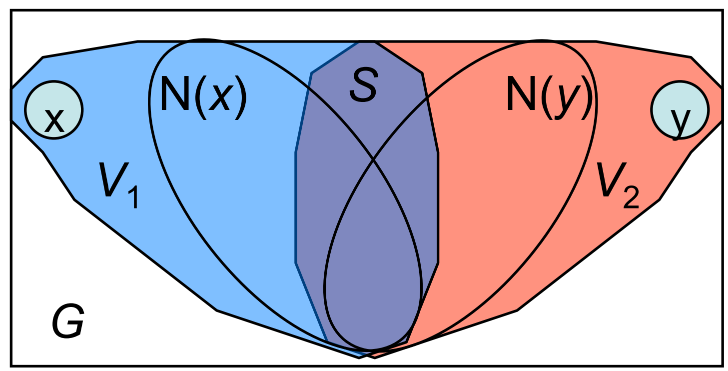
V_2=vertices of all S,y-paths

Properties of V_1, V_2:

S=V_1Å V_2

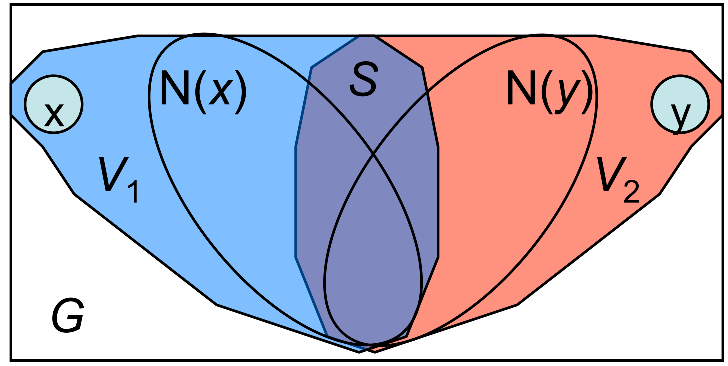
V_1Å (N(y)-S)=;

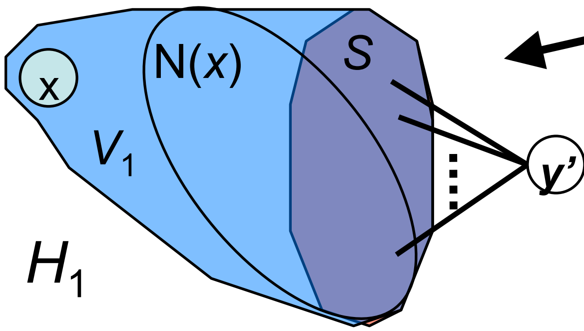
V_2Å (N(x)-S)=;

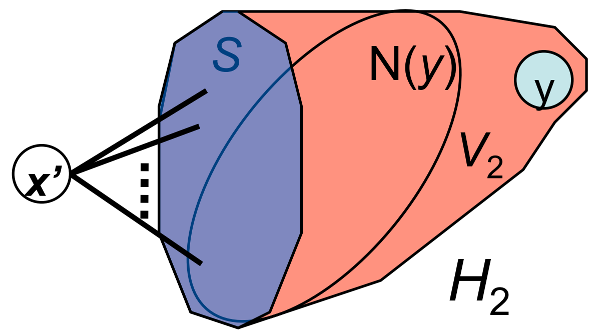


Menger’s Theorem (3)

From this decomposition generate new graphs H_1 and H_2.







Properties

$n(H_1), n(H_2) < n(G)$

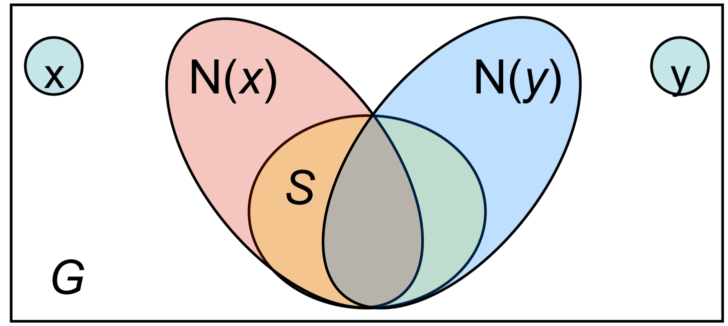
$\kappa_{H_1}(x,y') = \kappa_{H_2}(x',y) = k = |S|$

By induction, k pairwise internally disjoint x,y'-paths exist in H_1. Similarly for x',y-paths in H_2.

Piece together k pairwise internally disjoint x,y-paths in G.

Menger's Theorem (4)

<u>Case 2.</u> All minimum *x*,*y*-cuts *S* satisfy *S*µN(*x*)[N(*y*).



<u>Case 2A.</u> There exists some vertex *v*2{*x*}[N(*x*)[N(*y*)[{*y*}. Then *v* is in no minimum *x*,*y*-cut, and $\kappa(G\text{-}v)=k$. Induct on *G*-*v* to find $\kappa_{G\text{-}v}(x,y) = \lambda_{G\text{-}v}(x,y)=k$.

<u>Case 2B.</u> There exists some vertex *u*2N(*x*)\N(*y*). Then *u* must be in every minimum *x*,*y*-cut in order to separate *x* from *y*.

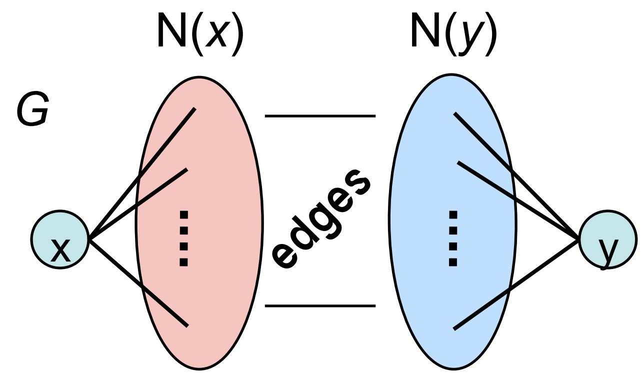
Therefore $\kappa(G\text{-}u)=k\text{-}1$.

Induct on *G*-*u* to find $\kappa_{G\text{-}u}(x,y) = \lambda_{G\text{-}u}(x,y)=k\text{-}1$.

Observe that the path *x*,*u*,*y* exists in *G* but not in *G*-*u*.

Menger's Theorem (5)

<u>Case 2C.</u> All minimum *x*,*y*-cuts *S* satisfy *S*μN(*x*)[N(*y*), and N(*x*),N(*y*) ***partition*** V(*G*)-{*x*,*y*}.



The set of *x*,*y*-paths are in natural bijection with the set of edges between N(*x*) and N(*y*).

Therefore *S* is a ***vertex cover*** of G[N(*x*)[N(*y*)].

König-Egerváry gives a matching of size |*S*|=*k*.

The matching edges correspond to pairwise internally disjoint *x*,*y*-paths, and so $\kappa(x,y) = \lambda(x,y)=k$. □