Definition (Algebraic) Let $a \in E$, an extension field of F. We call a algebraic over F if

$$\exists\, f(x) \in F[x] - \{0\} \text{ with } f(a) = 0.$$

Not algebraic over F $\leftrightarrow$ transcendental over F

Definition (Algebraic/Transcendental Extension)

E, an extension field of F, is an algebraic extension provided $\forall a \in E$, a is algebraic over F; and is a transcendental extension provided $\exists\, a \in E$, a is transcendental over F.

Ex $\sqrt{2}$ algebraic over $\mathbb{Q}$
i algebraic over $\mathbb{Q}$
$\mathbb{C}$ algebraic over $\mathbb{R}$
e, π transcendental over $\mathbb{Q}$
$\mathbb{R}$ transcendental over $\mathbb{Q}$.

Unknown: is $e+\pi$ transcendental over $\mathbb{Q}$?

<u>Theorem 21.1</u> Let $a \in E$, an extension of F.

a transcendental over F $\Rightarrow$ $F(a) \approx F(x)$.

a algebraic over F $\Rightarrow$ $F(a) \approx F[x]/\langle p(x)\rangle$, where
$p(x) = \arg\min \{\deg(f(x)) \mid f(x) \in F[x], f(a)=0\}$
and $p(x)$ is irreducible over F.

<u>Proof</u> $\left(F(x) = \{ f(x)/g(x) \mid f(x), g(x) \in F[x], g \neq 0\} \right)$

a transcendental over F:

$\varphi : F[x] \to F(a)$
$\varphi(f)$ defined by $f(x) \to f(a)$
has $\operatorname{Ker}\varphi = \{0\}$. Then so does
$\overline{\varphi} : F(x) \to F(a)$
$f(x)/g(x) \xrightarrow{\overline{\varphi}} f(a)/g(a)$ (Exercise: $\overline{\varphi}$ is onto)

a algebraic over F:
$\operatorname{Ker}\varphi \neq \{0\}$. $\operatorname{Ker}\varphi$ an ideal and $F[x]$
a PID $\Rightarrow$ $\operatorname{Ker}\varphi = \langle p(x)\rangle$ for some $p(x) \in F[x]$.
Thm 16.4: $p(x)$ has min degree of $f(x) \in \operatorname{Ker}\varphi$.
$p(a) = 0$ by defn. of φ.
$p(x)$ is irreducible, or else min degree is violated. $\square$

$F[x]/\langle p(x)\rangle \approx F(a)$
$(a \leftrightarrow x + \langle p(x)\rangle)$

Recall Ex 2 Chapt. 18:

In an integral domain, a and b are associates iff $\langle a \rangle = \langle b \rangle$.

Now consider $\langle p(x) \rangle$ with $\deg p(x) = n$ and

$$p(x) = p_n x^n + p_{n-1} x^{n-1} + \cdots + p_1 x + p_0 .$$

$p_n^{-1} p(x)$ is monic, and $\langle p(x) \rangle = \langle p_n^{-1} p(x) \rangle$

Every $f(x) \in F[x]$ with $\langle f(x) \rangle = \langle p(x) \rangle$ is an associate of $p_n^{-1} p(x)$. But the only monic associate is $p_n^{-1} p(x)$ itself.

Consequently,

Theorem 21.2 If a is algebraic over a field F, then there is a unique monic irreducible polynomial $p(x) \in F[x]$ with $p(a) = 0$.

Call this $p(x)$ the minimal polynomial for a over F.

Theorem 21.3 Divisibility Property

Let a be algebraic over F with minimal polynomial $p(x)$ over F. Then $p(x) \mid f(x)$ for all $f(x) \in F[x]$ with $f(a) = 0$.

Proof

These choices for $f(x)$ are in $\langle p(x) \rangle$ by proof of Thm. 21.1.

And $\langle p(x) \rangle$ is a principal ideal. $\square$

Degree of an Extension

Example 5, Chapter 19: an extension field E of F is a vector space over F.

Punted: every vector space has a basis, and thus a dimension, over its field.

Definition E an extension field of F

$$[E:F] := \deg_F E$$

$$= \begin{cases} n & \text{" finite extension "} \\ \infty & \text{" infinite extension"} \end{cases}.$$

Examples

$$\begin{array}{c|c|c}
E=\mathbb{Q}(\sqrt{2}) & \mathbb{Q}(i) & \mathbb{Q}(\sqrt[6]{2}) \\
\big|\, 2 & \big|\, 2 & \big|\, 6 \\
F=\mathbb{Q} & \mathbb{Q} & \mathbb{Q} \\
p(x)=x^2-2 & p(x)=x^2+1 & p(x)=x^6-2
\end{array}$$

$$\begin{array}{c|c|c}
\mathbb{Q}(\sqrt{3}) & \mathbb{Q}(\sqrt{5}) & \mathbb{Q}(\sqrt{3}+\sqrt{5}) \\
\big|\, 2 & \big|\, 2 & \big|\, 4 \\
\mathbb{Q} & \mathbb{Q} & \mathbb{Q} \\
p(x)=x^2-3 & p(x)=x^2-5 & p(x)=x^4-16x+4
\end{array}$$

Theorem 21.4 If E is a finite extension of F, then E is an algebraic extension of F.

Proof

Let $[E:F]=n\in\mathbb{Z}^+$.

(Any $n+1$ elements of E are linearly dependent over F.)

Let $a\in E$.

$\{a^n, a^{n-1}, \ldots, a, 1\}$ lin. dep. over F.

$\exists\ c_n, \ldots, c_1, c_0$ with $\sum_{i=0}^{n} c_i a^i = 0$.

WLOG, $c_n = 1$.

Then the minimal polynomial for a is

$p(x) = x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0$. $\square$

Theorem 21.5 Let K be a finite extension of E, and E be a finite extension of F. Then K is a finite extension of F with

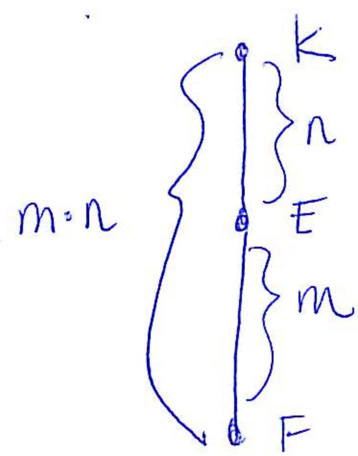
$$[K:F] = [K:E][E:F].$$

Proof sketch

Let $X = \{x_1, \dots, x_n\}$ be a basis for K over E.

Let $Y = \{y_1, \dots, y_m\}$ be a basis for E over F.

Then $YX = \{y_j x_i \mid 1 \le j \le m,\ 1 \le i \le n\}$ is a basis for K over F.



$[K:E] = n$

$[E:F] = m$

$[K:F] = m \cdot n$

Other Chapter 21 results

Let a, b be algebraic over F, a field with characteristic 0.
Then $\exists\, c \in F(a,b)$ with

$$F(a,b) = F(c).$$

Thm. 21.6

Thm 21.7 K an algebraic extension over E and E an algebraic extension over $F \Rightarrow K$ an algebraic extension over F.

Corollary The set of elements of an extension E of F that are algebraic over F is a subfield of E.

$$E' = \{a \in E \mid a \text{ algebraic over } F\} \subseteq E \quad \text{(as subfield)}.$$