Theorem 16.2 Division Algorithm

F a field, $f(x), g(x) \in F[x]$, $g(x) \neq 0$.

$\exists !$ $q(x), r(x) \in F[x]$ with

$$f(x) = g(x)q(x) + r(x),$$

where $r(x) = 0$ or $\deg r(x) < \deg g(x)$.

Proof sketch

Let $f(x) = \sum_{i=0}^{m} a_i x^i$, $g(x) = \sum_{j=0}^{n} b_j x^j$

and assume $\underset{m}{\underset{\shortparallel}{\deg f(x)}} \geq \underset{n}{\underset{\shortparallel}{\deg g(x)}}$.

$$f(x) = a_m x^m + a_{m-1} x^{m-1} + \cdots$$

$$g(x) = b_n x^n + b_{n-1} x^{n-1} + \cdots$$

note that $a_m, b_n \neq 0 \Rightarrow \frac{a_m}{b_n}$ exists.

$$\frac{a_m}{b_n} x^{m-n} g(x) = a_m x^m + \frac{a_m b_{n-1}}{b_n} x^{m-1} \cdots$$

and $f(x) - \frac{a_m}{b_n} x^{m-n} g(x)$ has smaller degree than $f(x)$. Apply induction on degree. (Uniqueness also needed.)

Cor. 1 Remainder Theorem

F a field, $f(x) \in F[x]$, $a \in F$. Then $f(a)$ is the remainder of the division of $f(x)$ by $x-a$.

Proof Write

$$f(x) = (x-a)q(x) + r(x)$$

uniquely by the division algorithm.

Note either $r(x) = 0$ or $\deg r(x) = 0$, so $r(x)$ is constant.