<u>Theorem 15.2</u> Let $\varphi : R \to S$ be a ring homomorphism. Then $\ker\varphi = \{r \in R \mid \varphi(r) = 0\}$ is an ideal of R.

<u>Proof</u>

Ker φ nonempty: $0 \in \ker\varphi$ since

φ is a group hom.

$$\varphi(0) = \varphi(0+0) = \varphi(0) + \varphi(0)$$
$$0_S = \varphi(0_R).$$

Property 1: Assume $a, b \in \ker\varphi$.

$$\begin{aligned}
\varphi(a-b) &= \varphi(a+(-b)) && \text{(Defn. "$-$")}\\
&= \varphi(a) + \varphi(-b) && \text{(Homom.)}\\
&= 0 + \varphi(-b) && (a \in \ker\varphi)\\
&= \varphi(-1\cdot b) && \text{(Thm. 12.1.5)}\\
&= -1\cdot\varphi(b) && \text{(Thm. 15.1)}\\
&= -1\cdot 0 && (b \in \ker\varphi)\\
&= 0.
\end{aligned}$$

$\therefore a-b \in \ker\varphi$.

Property 2: Assume $a \in \ker\varphi$ and $r \in R$.

$$\begin{aligned}
\varphi(ar) &= \varphi(a)\varphi(r) && \text{(Homom.)}\\
&= 0\cdot\varphi(r) && (a \in \ker\varphi)\\
&= 0 && \text{(Defn 0.)}
\end{aligned}$$

$\therefore ar \in \ker\varphi$.

By the Ideal test, $\ker\varphi$ is an ideal of R. $\square$

Theorem 15.3 First Isomorphism Theorem for Rings

Let $\varphi: R \to S$ be a ring homomorphism. Then

$$R/\ker\varphi \cong \varphi(R)$$

via the mapping $r + \ker\varphi \mapsto \varphi(r)$.

Proof As groups, $\ker\varphi \triangleleft R$ and $R/\ker\varphi \cong \varphi(R)$ by this mapping, which is a bijection. It remains to prove that the mapping preserves multiplication.

Let $r + \ker\varphi,\ s + \ker\varphi \in R/\ker\varphi$.

$$(r + \ker\varphi)(s + \ker\varphi) = rs + \ker\varphi \mapsto \varphi(rs)$$

where $\varphi(rs) = \varphi(r)\varphi(s)$

But by construction

$$r + \ker\varphi \mapsto \varphi(r)$$

and

$$s + \ker\varphi \mapsto \varphi(s).$$ $\square$

Theorem 15.4 Every ideal of a ring R is the kernel of a ring homomorphism of R. In particular, an ideal A is the kernel of the mapping $r \to r + A$ from R to R/A.

Proof

Define $\gamma : R \to R/A$
by $\gamma(r) = r + A$.

$R \to R/A$
$r \to r+A$
is the natural group hom.

Let $r, s \in R$.
Then $\gamma(r+s) = r+s+A$
$= r+A + s+A$
$= \gamma(r) + \gamma(s)$.

$\gamma(rs) = rs + A$
$= (r+A)(s+A)$
$= \gamma(r)\gamma(s)$.

$\ker \gamma = \{ r \in R \mid \gamma(r) = 0 \}$.
$\gamma(r) = r + A = 0$
iff $r \in A$.

So $\ker \gamma = A$. $\square$