**Theorem 15.2** Let \( \varphi : R \to S \) be a ring homomorphism. Then \( \ker \varphi = \{ r \in R \mid \varphi(r) = 0 \} \) is an ideal of \( R \).

**Proof**

**Ker \( \varphi \) nonempty:** \( 0 \in \ker \varphi \) since

\[
\varphi(0) = \varphi(0 + 0) = \varphi(0) + \varphi(0) = \varnothing_f(R)
\]

Property 1: Assume \( a, b \in \ker \varphi \).
\[
\begin{align*}
\varphi(a - b) &= \varphi(a) + \varphi(-b) \quad \text{(Defn. \( + \))} \\
&= \varphi(a) + \varphi(-b) \quad \text{(Homom.)} \\
&= 0 + \varphi(-b) \quad \text{(a \in Ker \( \varphi \))} \\
&= \varphi(-1 \cdot b) \quad \text{(Thm. 12.1.5)} \\
&= -1 \cdot \varphi(b) \quad \text{(Thm. 15.1)} \\
&= -1 \cdot 0 \quad \text{(b \in ker \( \varphi \))} \\
&= 0 \\
\therefore a - b \in \ker \varphi.
\end{align*}
\]

Property 2: Assume \( a \in \ker \varphi \) and \( r \in R \).
\[
\begin{align*}
\varphi(ar) &= \varphi(a) \cdot \varphi(r) \quad \text{(Homom.)} \\
&= 0 \cdot \varphi(r) \quad \text{(a \in Ker \( \varphi \))} \\
&= 0 \quad \text{(Defn. 0.)} \\
\therefore ar \in \ker \varphi.
\end{align*}
\]

By the ideal test, \( \ker \varphi \) is an ideal of \( R \). \( \square \)
Theorem 15.3  First Isomorphism Theorem for Rings

Let $\psi: R \rightarrow S$ be a ring homomorphism. Then

$$R / \ker \psi \cong \psi(R)$$

via the mapping $r + \ker \psi \mapsto \psi(r)$.

Proof  As groups, $\ker \psi \leq R$ and $R / \ker \psi \cong \psi(R)$
by this mapping, which is a bijection.
It remains to prove that the mapping preserves multiplication.

Let $r + \ker \psi, s + \ker \psi \in R / \ker \psi$.

$$(r + \ker \psi)(s + \ker \psi) = rs + \ker \psi \mapsto \psi(rs)$$

where $\psi(rs) = \psi(r) \psi(s)$.

But by construction

$r + \ker \psi \mapsto \psi(r)$
and
$s + \ker \psi \mapsto \psi(s)$.

$\square$
Theorem 15.4 Every ideal of a ring \( R \) is the kernel of a ring homomorphism of \( R \). In particular, an ideal \( A \) is the kernel of the mapping \( r \mapsto r + A \) from \( R \) to \( R/A \).

*Proof*

Define \( \gamma : R \to R/A \) by \( \gamma(r) = r + A \).

Let \( r, s \in R \).

Then \( \gamma(r + s) = (r + A) + (s + A) = \gamma(r) + \gamma(s) \).

\[
\gamma(rs) = rs + A = (r + A)(s + A) = \gamma(r) \gamma(s).
\]

Thus \( \ker \gamma = \{ r \in R \mid \gamma(r) = 0 \} \).

If \( \gamma(r) = r + A = 0 \) iff \( r \in A \).

So \( \ker \gamma = A \).