Theorem 14.1 Ideal Test

A nonempty subset A of a ring R is an ideal of R if

1. $a-b \in A$ whenever $a, b \in A$
2. ra and ar are in A whenever $a \in A$ and $r \in R$.

Proof If we can show A with these properties is a subring, then we are done since 2. would then give that A is an ideal.

By Theorem 12.3 we need:

A nonempty assumed in Thm 14.1 premise.

$a, b \in \Rightarrow a-b \in A$ assumed in Thm 14.1 property 1.

$a, b \in A \Rightarrow ab \in A$ Let $a, b \in A$.

$b \in A \Rightarrow b \in R$.
By Thm 14.1 property 2,
$ab \in A$. $\square$

Theorem 14.3

Let R be a commutative ring with unity, and Let A be an ideal of R. Then

R/A is an integral domain iff A is prime.

Proof

($\Rightarrow$) Assume R/A is an integral domain.

Let $ab \in A$. $(a, b \in R)$

Note $(a+A)(b+A) = ab + A = 0 + A$.

R/A an integral domain $\Rightarrow$

$$(a+A) = 0 + A$$

or $(b+A) = 0 + A$.

Thus $a \in A$

or $b \in A$, respectively, and A is prime.

($\Leftarrow$) Assume A is prime.

R commutative $\Rightarrow$ R/A commutative.

R has unity $1 \Rightarrow R/A$ has unity $1 + A$,

since $(1+A)(r+A) = r + A = (r+A)(1+A)$.

Now let $a+A, b+A \in R/A$ and assume

$(a+A)(b+A) = 0+A$

This means in the factor ring language

$$(a+A)(b+A) = ab + A = 0 + A$$

or $ab \in A$.

A prime $\Rightarrow$ $a \in A$

or $b \in A$;

Thus $a + A = 0 + A$

or $b + A = 0 + A$ respectively. $\square$

Theorem 14.4 Let R be a commutative ring with unity and let A be an ideal of R. Then R/A is a field iff A is maximal.

Proof

($\Rightarrow$) Assume R/A is a field.

Let B be an ideal with $A \subset B$ (proper subset).

Then $\exists\, b \in B - A$, and $b + A \neq 0 + A$.

R/A is a field, so $\exists\, c + A$ with

$$(b+A)(c+A) = 1 + A \quad \} \text{ unity in } R/A$$

$$bc + A = 1 + A$$

thus $1 - bc \in A \subset B$.

$$\underbrace{1 - bc}_{\in B} + \underbrace{bc}_{\in B} = 1 \in B$$

And so $B = R$, and A is maximal.

($\Leftarrow$) Assume A is maximal.

As before, R/A is a commutative ring with unity $1 + A$.

Now let $b + A \in R/A$ where $b \notin A$.

Set $B = \{ br + a \mid r \in R,\ a \in A \}$.

(Exercise) B is an ideal of R, and $A \subset B$.

Thus $B = R$, and in particular, $1 \in B$.

Therefore $1 = bc + a'$, for some $c \in R, a' \in A$.

$$1 + A = bc + a' + A = bc + A$$
$$= (b+A)(c+A).$$

Thus $b + A$ is a unit. $\square$