Math 430 (1)
Ch 10 Group Homomorphisms

Theorem 10.1.1. $\varphi(e) = \bar{e}$

Proof

$$e = e \cdot e$$

$$\varphi(e) = \varphi(e \cdot e) = \varphi(e)\varphi(e) \quad \text{Property 2}$$

$$\bar{e} = \varphi(e) \quad \text{Left cancellation.} \ \square$$

Theorem 10.1.2 $\varphi(g^n) = (\varphi(g))^n \ \forall n \in \mathbb{Z}$.

Proof Sketch

$n \in \mathbb{Z}^+$

$$\varphi(g^n) = \varphi(g^{n-1}g) = \varphi(g^{n-1})\varphi(g) \quad \text{Prop. 2.}$$

$$= \cdots = (\varphi(g))^n \quad \text{induction.}$$

$n \notin \mathbb{Z}^+$: same as Theorem 6.2 $\square$

Theorem 10.1.3 $|g| < \infty \Rightarrow |\varphi(g)| \mid |g|$.

Proof Let $|g| = n \in \mathbb{Z}^+$.

$$(\varphi(g))^n = \varphi(g^n) \quad \text{by Part 2.}$$

$$= \varphi(e) \quad \text{by } |g| = n$$

$$= \bar{e} \quad \text{by Part 1.}$$

$\therefore |\varphi(g)| \mid n$ by Cor 2 to Thm 4.1 $\square$
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Theorem 10.1.6 If $\varphi(g) = g'$, then

$\varphi^{-1}(g') = \{x \in G \mid \varphi(x) = g'\} = g \operatorname{Ker} \varphi$.

Proof Assume $\varphi(g) = g'$.

$(\varphi^{-1}(g') \subseteq g \operatorname{Ker} \varphi)$

Let $x \in \varphi^{-1}(g')$.

| | | |
|---|---|---|
| Then | $\varphi(x) = g' = \varphi(g)$ | by assumptions |
| | $\varphi(x)\varphi(g)^{-1} = \bar{e}$ | by cancellation |
| | $\varphi(x)\varphi(g^{-1}) = \bar{e}$ | by Part 2 |
| | $\varphi(xg^{-1}) = \bar{e}$ | by Hom. Property |
| | $xg^{-1} \in \operatorname{Ker}\varphi$ | by defn. Ker $\varphi$ |
| | $x \in g\operatorname{Ker}\varphi$ | p 138 part 4. |

$(g \operatorname{Ker}\varphi \subseteq \varphi^{-1}(g'))$

Let $x \in g \operatorname{Ker}\varphi$.

Then $x = gk$ for some $k \in \operatorname{Ker}\varphi$.

| | |
|---|---|
| $\varphi(x) = \varphi(gk) = \varphi(g)\varphi(k)$ | by Hom. Prop. |
| $= \varphi(g)\bar{e}$ | by defn. Ker $\varphi$ |
| $= g'$ | by assumption. |

$\therefore \quad x = gk \in \varphi^{-1}(g')$. $\square$

## Theorem 10.2.1-3

1. $H \leq G \Rightarrow \varphi(H) \leq \overline{G}$
2. $H$ cyclic $\Rightarrow \varphi(H)$ cyclic.
3. $H$ Abelian $\Rightarrow \varphi(H)$ Abelian.

Proof see Theorem 6.3. Relies only on property 2.

## Theorem 10.2.4 $H \triangleleft G \Rightarrow \varphi(H) \triangleleft \varphi(G)$.

Proof (direct)

Assume $H \triangleleft G$.

Let $x \in \varphi(H)$ and $y \in \varphi(G)$.

Then there exist $h \in H$ and $g \in G$ with

$$\varphi(h) = x \quad \text{and} \quad \varphi(g) = y.$$

$$\begin{aligned} yxy^{-1} &= \varphi(g)\varphi(h)\varphi(g)^{-1} \\ &= \varphi(ghg^{-1}) \end{aligned}$$

$H \triangleleft G \Rightarrow ghg^{-1} \in H$.

Thus $\varphi(ghg^{-1}) \in \varphi(H)$.

By Normal Subgroup Test, $\varphi(H) \triangleleft \varphi(G)$.

$\square$

**Theorem 10.2.5** If $|\mathrm{Ker}\,\varphi| = n$, then $\varphi$ is an $n$-to-1 mapping from $G$ onto $\varphi(G)$.

**Proof** By definition of $\varphi(G)$, $\varphi$ is onto $\varphi(G)$.

Let $x, y \in \varphi(G)$, and suppose $x \neq y$.

There exist $a, b \in G$ with $x = \varphi(a)$, $y = \varphi(b)$.

By 10.1.6,
$$\varphi^{-1}(x) = a\,\mathrm{Ker}\,\varphi$$
$$\varphi^{-1}(y) = b\,\mathrm{Ker}\,\varphi$$

By Page 138 part 5,
$$|\varphi^{-1}(x)| = |a\,\mathrm{Ker}\,\varphi| = |b\,\mathrm{Ker}\,\varphi| = |\varphi^{-1}(y)|$$
$$= |\mathrm{Ker}\,\varphi|$$
$$= n \quad \text{by assumption.}$$

Thus the preimage of each $g' \in \varphi(G)$ is a distinct set of $n$ elements of $G$. $\square$