Theorem 10.1.1 \( \varphi(e) = e \)

**Proof**

\[
e = e \cdot e
\]
\[
\varphi(e) = \varphi(e \cdot e) = \varphi(e) \varphi(e) \quad \text{Property 2}
\]
\[
\bar{e} = \varphi(e)
\]

*Left cancellation.* \( \Box \)

Theorem 10.1.2 \( \varphi(g^n) = (\varphi(g))^n \neq \begin{array}{l} n \in \mathbb{Z} \end{array} \)

**Proof Sketch**

\[
\varphi(g^n) = \varphi(g^{-1} g) = \varphi(g^{-1}) \varphi(g) \quad \text{Prop 2.}
\]
\[
\begin{array}{l}
\begin{array}{l}
\text{induction.} \\
\text{n \in \mathbb{Z}^+} \\
\end{array}
\end{array}
\]

\[
\begin{array}{l}
\begin{array}{l}
n \notin \mathbb{Z}^+: \text{ same as Theorem 6.2} \quad \Box
\end{array}
\end{array}
\]

Theorem 10.1.3 \( |g| < \infty \Rightarrow |\varphi(g)| | |g| \)

**Proof** Let \( |g| = n \in \mathbb{Z}^+ \).

\[
(\varphi(g))^n = \varphi(g^n) \quad \text{by Part 2.}
\]
\[
= \varphi(e) \quad \text{by \( |g| = n \)}
\]
\[
= \bar{e} \quad \text{by Part 1.}
\]

\[ \therefore \ |\varphi(g)| | |g| \quad \text{by Cor 2 to Thm 4.1 } \Box \]

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Theorem 10.1.6 If $\ell(g') = g'$, then

$\ell^{-1}(g') = \{x \in G \mid \ell(x) = g'\} = g' \ker \ell$.

Proof Assume $\ell(g') = g'$.

( $\ell^{-1}(g') \leq g \ker \ell$ )

Let $x \in \ell^{-1}(g')$.

Then $\ell(x) = g' = \ell(g)$

by assumptions

$\ell(x) \ell(g)^{-1} = \bar{e}$

by cancellation

$\ell(x) \ell(g^{-1}) = \bar{e}$

by Part 2

$\ell(xg^{-1}) = \bar{e}$

by Hom. Property

$xg^{-1} \in \ker \ell$

$p 138$ part 4.

( $g \ker \ell \leq \ell^{-1}(g')$ )

Let $x \in g \ker \ell$.

Then $x = gk$ for some $k \in \ker \ell$.

$\ell(x) = \ell(gk) = \ell(g) \ell(k)$

by Hom. Prop.

$= \ell(g) \bar{e}$

by defn. $\ker \ell$

$= g'$

by assumption.

$\therefore x = gk \in \ell^{-1}(g')$.  \qed
Theorem 10.2.1-3

1. \( H \leq G \implies \mathcal{U}(H) \leq \mathcal{U}(G) \)
2. \( H \) cyclic \( \implies \mathcal{U}(H) \) cyclic.
3. \( H \) Abelian \( \implies \mathcal{U}(H) \) Abelian.

Proof see Theorem 6.3. Relies only on property 2.

Theorem 10.2.4 \( H \triangleleft G \implies \mathcal{U}(H) \triangleleft \mathcal{U}(G) \).

Proof (direct)
Assume \( H \triangleleft G \).
Let \( x \in \mathcal{U}(H) \) and \( y \in \mathcal{U}(G) \).
Then there exist \( h \in H \) and \( g \in G \) with \( \mathcal{U}(h) = x \) and \( \mathcal{U}(g) = y \).
\[ yxy^{-1} = \mathcal{U}(g) \mathcal{U}(h) \mathcal{U}(g)^{-1} \]
\[ = \mathcal{U}(ghg^{-1}) \]
\( H \triangleleft G \implies ghg^{-1} \in H. \)
Thus \( \mathcal{U}(ghg^{-1}) \in \mathcal{U}(H). \)
By Normal Subgroup Test, \( \mathcal{U}(H) \triangleleft \mathcal{U}(G). \)
\( \square \)
Theorem 10.2.5: If $|\ker \psi| = n$, then $\psi$ is an $n$-to-$1$ mapping from $G$ onto $\psi(G)$.

**Proof:** By definition of $\psi(G)$, $\psi$ is onto $\psi(G)$.

Let $x, y \in \psi(G)$, and suppose $x \neq y$.

There exist $a, b \in G$ with $x = \psi(a)$, $y = \psi(b)$.

By 10.1.6, $\psi^{-1}(x) = a \ker \psi$

$\psi^{-1}(y) = b \ker \psi$

By page 138 part 5,

$|\psi^{-1}(x)| = |a \ker \psi| = |b \ker \psi| = |\psi^{-1}(y)|$

$= |\ker \psi|$

$= n$ by assumption.

Thus the preimage of each $g' \in \psi(G)$ is a distinct set of $n$ elements of $G$.

\[ G \xrightarrow{\psi} \psi(G) \xrightarrow{\psi^{-1}} G \]