Theorem 10.2.8 $\overline{K} \lhd \overline{G} \Rightarrow \varphi^{-1}(\overline{K}) \lhd G$.

Proof By Normal Subgroup Test.

Let $x \in G$ and $k \in \varphi^{-1}(\overline{K})$.

Then $\varphi(k) \in \overline{K}$.

$$\varphi(xkx^{-1}) = \varphi(x)\varphi(k)(\varphi(x))^{-1}$$

and $\overline{K} \lhd \overline{G}$, so

$$\varphi(xkx^{-1}) \in \overline{K}.$$

But then $xkx^{-1} \in \varphi^{-1}(\overline{K})$.

Therefore $\varphi^{-1}(\overline{K}) \lhd G$. $\square$

Cor. $\operatorname{Ker}\varphi \lhd G$. set $\overline{K} = \{\overline{e}\}$

$\operatorname{Ker}\varphi = \varphi^{-1}(\overline{e})$. Apply 10.2.8

$\Rightarrow \operatorname{Ker}\varphi \lhd G$ $\square$

"Kernels are normal"

Thm 10.3 First Isomorphism Theorem

Let φ be a group homomorphism from G to $\overline{G}$.
Then the mapping $\psi : G/\mathrm{Ker}\,\varphi \to \varphi(G)$
defined by $\psi(g\,\mathrm{Ker}\,\varphi) = \varphi(g)$
is an isomorphism, so that $G/\mathrm{Ker}\,\varphi \approx \varphi(G)$.

<u>Proof</u>

ψ well-defined: Suppose $a\,\mathrm{Ker}\,\varphi = b\,\mathrm{Ker}\,\varphi$.

By 10.1.5, $\underset{\psi(a\,\mathrm{Ker}\,\varphi)}{\overset{\shortparallel}{\varphi(a)}} = \underset{\psi(b\,\mathrm{Ker}\,\varphi)}{\overset{\shortparallel}{\varphi(b)}}$.

ψ 1-1:

Suppose $\psi(a\,\mathrm{Ker}\,\varphi) = \psi(b\,\mathrm{Ker}\,\varphi)$.
By defn., $\varphi(a) = \varphi(b)$
By 10.1.5, $a\,\mathrm{Ker}\,\varphi = b\,\mathrm{Ker}\,\varphi$.

ψ onto: Let $y \in \varphi(G)$.
Then $y = \varphi(g)$ for some $g \in G$ by defn. $\varphi(G)$
$y = \varphi(g) = \psi(g\,\mathrm{Ker}\,\varphi)$ by defn ψ.

ψ op. pres.: Let $x\,\mathrm{Ker}\,\varphi,\ y\,\mathrm{Ker}\,\varphi \in G/\mathrm{Ker}\,\varphi$.

$$\psi(x\,\mathrm{Ker}\,\varphi\; y\,\mathrm{Ker}\,\varphi) = \psi(xy\,\mathrm{Ker}\,\varphi) = \varphi(xy)$$
$$= \varphi(x)\varphi(y) = \psi(x\,\mathrm{Ker}\,\varphi)\,\psi(y\,\mathrm{Ker}\,\varphi). \quad \square$$

$$\begin{tikzcd}
G \arrow[rr, "\varphi"] \arrow[dr, "\gamma"'] & & \varphi(G) \\
& G/\mathrm{Ker}\,\varphi \arrow[ur, "\psi"'] &
\end{tikzcd}$$

Commutative diagram

$\gamma : G \to G/\mathrm{Ker}\,\varphi$
$\gamma(g) = g\,\mathrm{Ker}\,\varphi$
"natural homomorphism"

$\varphi = \psi \circ \gamma$

Ex 15 N/C Theorem Let $H \leq G$.

Normalizer $N(H) = \{ x \in G \mid xHx^{-1} = H \}$

Centralizer $C(H) = \{ x \in G \mid xhx^{-1} = h \text{ for all } h \in H \}$

define
$$\Theta : N(H) \to \mathrm{Aut}(H)$$
$$\Theta(g) = \varphi_g \leftarrow \left(\begin{array}{c} \text{inner automorphism} \\ \varphi_g(h) = ghg^{-1} \end{array} \right)$$

Then $N(H)/C(H)$ is isomorphic to a subgroup of $\mathrm{Aut}(H)$.

<u>Proof</u> The kernel of Θ is all $g \in N(H)$ such that $\varphi_g = \varphi_e$; i.e.,

$$\varphi_g(h) = \varphi_e(h) \quad \text{for all } h \in H$$
$$ghg^{-1} = ehe^{-1} = h \quad \text{for all } h \in H.$$

By defn. this is $C(H)$.

Θ is operation-preserving since

$$\Theta(g_1 g_2) = \varphi_{g_1 g_2}$$
$$\varphi_{g_1 g_2}(h) = g_1 g_2 h (g_1 g_2)^{-1} = g_1 g_2 h g_2^{-1} g_1^{-1}$$
$$= \varphi_{g_1} \varphi_{g_2}(h)$$

So $\Theta(g_1 g_2) = \varphi_{g_1} \varphi_{g_2}$.

Apply Thm 10.3 to get $N(H)/C(H) \simeq \Theta(N(H))$ and 10.2.1 to get $N(H)/C(H) \approx \Theta(N(H)) \leq \mathrm{Aut}(H)$.

$\square$

Thm 10.4 Normals are Kernel.

If $N \triangleleft G$ then $\gamma : G \to G/N$
defined by $\gamma(g) = gN$
is a homomorphism with kernel N.

Proof γ is operation preserving:

$$\gamma(g_1 g_2) = g_1 g_2 N = g_1 N g_2 N$$
$$= \gamma(g_1)\gamma(g_2).$$

$$\operatorname{Ker}\gamma = \gamma^{-1}(N) = \{g \in G \mid \gamma(g) = N\}.$$

Now $\gamma(g) = gN = N$
precisely when $g \in N$ by p138 part 2. $\square$