Definition. A field E is an extension field of a field F if $F \subseteq E$ and the operations of F are those of E restricted to F.

Theorem: Fundamental Theorem of Field Theory (Kronecker's Theorem, 1887). Let F be a field and let $f(x)$ be a nonconstant polynomial in $F[x]$. Then there is an extension field E of F in which $f(x)$ has a zero.

(1) Finding field extensions.
(a) Find an extension field E of $\mathbb{Q}$ in which $f(x) = x^2 - 2$ has a zero. (Do not express the field as $\mathbb{Q}[\alpha]$).
(b) Which field of the form $\mathbb{Q}[\alpha]$ is E isomorphic to?
(c) Does $f(x)$ factor into linear factors over E?

(1) Finding field extensions, from GA 17C (3).
(a) Find an extension field E of $\mathbb{Z}_2$ in which $f(x) = x^3 + x + 1$ has a zero. (Do not express the field as $\mathbb{Z}[\alpha]$).
(b) How many elements are there in E?

(3) Finding field extensions.
(a) Find an extension field E of $\mathbb{R}$ in which $f(x) = (x^5 - 1)/(x - 1)$ has a zero. (Do not express the field as $\mathbb{R}[\alpha]$).
(b) Conjecture which field of the form $\mathbb{R}[\alpha]$ that E is isomorphic to.
(c) Conjecture as to whether $f(x)$ factors into linear factors over E.

Definition. Let E be an extension field of F and let $f(x) \in F[x]$. We say that $f(x)$ *splits* in E if $f(x)$ can be factored as a product of linear factors in $E[x]$. We call E and *splitting field for* $f(x)$ *over* f if $f(x)$ splits in E but in no proper subfield of E.

Properties of a splitting field E of $f(x)$ over F.
A splitting field E is
(i) with respect to a polynomial $f(x)$,
(ii) with respect to a field F with $f(x) \in F[x]$, and
(iii) Satisfies $E = \cap_{E'} E'$ where E' ranges over all fields containing F over which $f(x)$ factors into linear factors.

(4) What is the splitting field of $f(x) = x^2 - 2$ over $\mathbb{Q}$? (See (1).)

(5) Constructing splitting fields.
(a) Let $f(x) \in F[x]$ be an irreducible nonconstant polynomial. Construct an extension field E in which $f(x)$ has a zero.
(b) Call the zero from (a) α. Use the Factor Theorem (Cor. 2, p.296) to factor out a linear factor of $f(x)$ over E. (If there are no irreducible factors of degree ≥ 2, we are done, and E is a splitting field of $f(x)$ over F. But assume this is not the case and go on to (c).)
(c) Using the form of the factorization of $f(x)$ you found in (b), Find an extension field E' of E over which $f(x)$ factors further.

Theorem 20.2: Existence of Splitting Fields. Let F be a field and let $f(x)$ be a nonconstant element of $F[x]$. Then there exists a splitting field E for $f(x)$ over F.

Theorem 20.3: $\mathbf{F(a) \approx F[x]/\langle p(x)\rangle}$. Let F be a field and let $p(x) \in F[x]$ be irreducible over F. If a is a zero of $p(x)$ in some extension E of F, then $F(a)$ is isomorphic to $F[x]/\langle p(x)\rangle$. Furthermore, if $\deg p(x) = n$, then every member of $F(a)$ can be uniquely expressed in the form

$$c_{n-1}a^{n-1} + c_{n-2}a^{n-2} + \cdots + c_1 a + c_0,$$

where $c_0, c_1, \ldots, c_{n-1} \in F$.

Corollary: $\mathbf{F(a) \approx F(b)}$. Let F be a field and let $p(x) \in F[x]$ be irreducible over F. If a is a zero of $p(x)$ in some extension E of F and b is a zero of $p(x)$ in some extension E' of F, then the fields $F(a)$ and $F(b)$ are isomorphic.