Definition. A set V is said to be a *vector space* over a field F if V is an Abelian group under addition (denoted by $+$), and, if for each $a \in F$ and $v \in V$, there is an element av in V such that the following conditions hold for all $a, b \in F$ and all $u, v \in V$.
1. $a(v+u) = av + au$
2. $(a+b)v = av + bv$
3. $a(bv) = (ab)v$
4. $1v = v$.
We call F the set of *scalars* and V the set of *vectors.*

Examples of Vector Spaces: $F[x]$ over the field F, $m \times n$ matrices over the field F, $\mathbb{C} = \{a + bi | a, b \in \mathbb{R}\}$ over the field $\mathbb{R}$, the field E over its subfield F. In all but the last example, there is an obvious placeholder for coordinate-wise vector addition, and for scalar multiplication.

Definition. Let V be a vector space over a field F, and let U be a subset of V. We say that U is a *subspace* of V if U is also a vector space over F under the operations of V.

Lemma: Subspace Test. A nonempty subset U of a vector space V over a field F is a subspace of V, if, for every $u, u \in U$ and every $a \in F$, $u + u' \in U$ and $au \in U$.

(1) Let $V = \mathbb{R}^3$ and $W = \{(a, b, c) \in V | a + b = c\}$. Show that W is a subspace of V.

Definition. A set S of vectors is said to be *linearly dependent* over the field F if there are vectors $v_1, v_2, \ldots, v_n$ from S and elements $a_1, a_2, \ldots, a_n$ from F, not all zero, such that $\sum_{i=1}^{n} a_i v_i = 0$. The set of vectors is called *linearly independent* if it is not linearly dependent.

Definition. Let V be a vector space over F. A subset B of V is called a *basis* for V if B is linearly independent over F, and every element of V is a linear combination of elements of B.

Theorem 19.1: Invariance of Basis Size. If $\{u_1, \ldots, u_m\}$ and $\{w_1, \ldots, w_n\}$ are both bases of a vector space V, then $m = n$.

(2) Give a basis for the subspace W in (1) and prove both the spanning and linear independence properties.

Definition. A vector space that has a basis consisting of n elements is said to have *dimension* n. The trivial vector space is defined to have dimension 0 and to be spanned by the empty basis $\emptyset$.

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(3) Let V be a vector space over F of dimension 5, and let U and W be subspaces of V both of dimension 3.
(a) Write down the form of a choice of bases B_U and B_W for U and W, respectively.
(b) Compare $B_U \cup B_W$ to $\dim V$; what is true as a result?
(c) Starting from a relation obtained in (b), show that $U \cap W \neq \{0\}$.

(4) Let B be a finite subset of a vector space V. We wish to prove that B is a basis for V iff every member of V is a unique linear combination of the elements of B.
(a) For the forward direction of the proof, what does the spanning property of bases give us?
(b) For the forward direction of the proof, suppose that B is a basis, but there is some $v \in V$ that can be obtained non-uniquely as a linear combination of the elements of B. What is the conclusion?
(c) For the reverse direction of the proof, why does the spanning property of B hold?
(d) How does the uniqueness of representing $0 \in V$ as a linear combination give us the linear independence of B?

Definition. Let V and W be vector spaces over F. A *linear transformation* from V to W is a mapping $T : V \to W$ satisfying
1. $T(u + v) = T(u) + T(v)$,
2. $T(au) = aT(U)$,
for all $u, v \in V$ and for all $a \in F$. If T is additionally a bijection, then T is a *vector space isomorphism.*

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(5) Let V be a vector space of dimension n over a field F. Show that V is (vector space-) isomorphic to $F^n = \{(a_1, a_2, \dots, a_n) | a_i \in F\}$ as follows:
(a) Recall the standard basis for F^n.
(b) Write down a basis for V.
(c) Define $T(v)$ for an arbitrary $v \in V$ in terms of the two bases selected in (a) and (b).
(d) Prove that T is a vector space isomorphism.