Definition. Let $D$ be an integral domain, and let $a, b, c \in D$.
If $a = ub$ for some unit $u \in D$, then $a$ and $b$ are associates.
If $a$ is a nonzero non-unit, and writing $a = bc$ implies $b$ or $c$ is a unit, $a$ is irreducible.
If $a$ is a nonzero non-unit, and $a|bc$ implies $a|b$ or $a|c$, then $a$ is prime.

Definition. Let $d \in \mathbb{Z}$ such that $d \neq 1$ and $d$ is not divisible by the square of a prime. Then $\mathbb{Z}[\sqrt{d}] := \{a + b\sqrt{d} | a, b \in \mathbb{Z}\}$ is a ring equipped with a norm function $N$ with the properties described in Lemma A.

Lemma A. The norm function $N : \mathbb{Z}[\sqrt{d}] \to \mathbb{N}$ defined by $N(a + b\sqrt{d}) = |a^2 - db^2|$ satisfies:
1. $N(x) = 0$ iff $x = 0$,
2. $N(xy) = N(x)N(y)$ for all $x, y$,
3. $x \in \mathbb{Z}[\sqrt{d}]$ is a unit iff $N(x) = 1$, and
4. For $x \in \mathbb{Z}[\sqrt{d}]$, if $N(x)$ is prime, then $x$ is irreducible in $\mathbb{Z}[\sqrt{d}]$.

(1) Prove Lemma A. Parts (1)-(2) are straightforward calculation. For part (3), recall the definition of a unit, and remember that $N$ is a nonnegative integer function. Part (4) relies on parts (2)-(3).

Theorem 18.1: Prime Implies Irreducible.
In an integral domain, every prime is an irreducible.

Theorem 18.2: PID Implies Irreducible Equals Prime.
In a principal ideal domain, an element is an irreducible iff it is a prime.

Lemma B. $\mathbb{Z}[x]$ is not a principal ideal domain.
(2) Trace through example 1: in \(\mathbb{Z}[\sqrt{-3}]\), \(1 + \sqrt{-3}\) is irreducible but not prime.
(a) Compute \(N(1 + \sqrt{-3})\). Suppose that \(1 + \sqrt{-3}\) can be written as \(xy\) for some \(x, y \in \mathbb{Z}[\sqrt{-3}]\). Use Lemma A(2) to write down the possibilities for \(N(x), N(y)\). Assuming neither \(x\) nor \(y\) is a unit, what possibilities for \(N(x), N(y)\) remain?
(b) Solve \(N(x) = 2\) for \(x\), by writing \(x\) in the form \(a + b\sqrt{-3}\). What is the conclusion?
(c) Verify the equation \((1 + \sqrt{-3})(1 - \sqrt{-3}) = 2 \cdot 2\), so that \((1 + \sqrt{-3})\) divides \(2 \cdot 2\).
(d) If \((1 + \sqrt{-3})\) is to be prime, then \((1 + \sqrt{-3})|2 \cdot 2\) must imply that \((1 + \sqrt{-3})|2\); in other words, there must be a solution to the equation \((1 + \sqrt{-3})(a + b\sqrt{-3}) = 2\). Try to solve this equation and draw a conclusion about whether \((1 + \sqrt{-3})\) is prime.

(3) Let \(a, b\) be elements of an integral domain. Prove that if \(b\) is nonzero, and \(a\) is not a unit, then \((ab)\) is a proper subset of \((b)\).