Definition. Let D be an integral domain, and let $a, b, c \in D$.
If $a = ub$ for some unit $u \in D$, then a and b are *associates*.
If a is a nonzero non-unit, and writing $a = bc$ implies b or c is a unit, a is *irreducible*.
If a is a nonzero non-unit, and $a|bc$ implies $a|b$ or $a|c$, then a is *prime*.

Definition. Let $d \in \mathbb{Z}$ such that $d \neq 1$ and d is not divisible by the square of a prime. Then $\mathbb{Z}[\sqrt{d}] := \{a + b\sqrt{d} | a, b \in \mathbb{Z}\}$ is a ring equipped with a norm function N with the properties described in Lemma A.

Lemma A. The norm function $N : \mathbb{Z}[\sqrt{d}] \to \mathbb{N}$ defined by $N(a + b\sqrt{d}) = |a^2 - db^2|$ satisfies:
(1) $N(x) = 0$ iff $x = 0$,
(2) $N(xy) = N(x)N(y)$ for all x, y,
(3) $x \in \mathbb{Z}[\sqrt{d}]$ is a unit iff $N(x) = 1$, and
(4) For $x \in \mathbb{Z}[\sqrt{d}]$, if $N(x)$ is prime, then x is irreducible in $\mathbb{Z}[\sqrt{d}]$.

(1) Prove Lemma A. Parts (1)-(2) are straightforward calculation. For part (3), recall the definition of a unit, and remember that N is a nonnegative integer function. Part (4) relies on parts (2)-(3).

Theorem 18.1: Prime Implies Irreducible.
In an integral domain, every prime is an irreducible.

Theorem 18.2: PID Implies Irreducible Equals Prime.
In a principle ideal domain, an element is an irreducible iff it is a prime.

Lemma B. $\mathbb{Z}[x]$ is not a principal ideal domain.

(2) Trace through example 1: in $\mathbb{Z}[\sqrt{-3}]$, $1+\sqrt{-3}$ is irreducible but not prime.
(a) Compute $N(1+\sqrt{-3})$. Suppose that $1+\sqrt{-3}$ can be written as xy for some $x, y \in \mathbb{Z}[\sqrt{-3}]$. Use Lemma A(2) to write down the possibilities for $N(x), N(y)$. Assuming neither x nor y is a unit, what possibilities for $N(x)$, $N(y)$ remain?
(b) Solve $N(x) = 2$ for x, by writing x in the form $a + b\sqrt{-3}$. What is the conclusion?
(c) Verify the equation $(1+\sqrt{-3})(1-\sqrt{-3}) = 2 \cdot 2$, so that $(1+\sqrt{-3})$ divides $2 \cdot 2$.
(d) If $(1+\sqrt{-3})$ is to be prime, then $(1+\sqrt{-3}) | 2 \cdot 2$ must imply that $(1+\sqrt{-3})|2$; in other words, there must be a solution to the equation $(1+\sqrt{-3})(a+b\sqrt{-3}) = 2$. Try to solve this equation and draw a conclusion about whether $(1+\sqrt{-3})$ is prime.

(3) Let a, b be elements of an integral domain. Prove that if b is nonzero, and a is not a unit, then $\langle ab \rangle$ is a proper subset of $\langle b \rangle$.