Theorem 17.2: Over $\mathbb{Q}$ implies over $\mathbb{Z}$.
Let $f(x) \in \mathbb{Z}[x]$. If $f(x)$ is reducible over $\mathbb{Q}$, then it is reducible over $\mathbb{Z}$.

(1) Produce the factorization of $f(x) = 6x^2 + x - 2 = (3x - 3/2)(2x + 4/3)$, where $g(x) = 3x - 3/2$ and $h(x) = 2x + 4/3$ as follows.
(a) Determine that $f(x)$ is primitive.
(b) Compute a, the lcm of the denominators of the coefficients of $g(x)$, and b, the lcm of the denominators of the coefficients of $g(x)$; write $abf(x) = ag(x)bh(x)$.
(c) Compute the contents c_1 of $ag(x)$ and c_2 of $bh(x)$, and find primitive polynomials $g_1(x)$ and $h_1(x)$ with $ag(x) = c_1 g_1(x)$ and $bh(x) = c_2 h_1(x)$.
(d) Use Gauss' Lemma to determine the content of $c_1 c_2 g_1(x) h_1(x)$, and deduce a factorization of $f(x)$ over $\mathbb{Z}[x]$.

Theorem 17.3: Mod p irreducibility test.
Let p be a prime and suppose that $f(x) \in \mathbb{Z}[x]$ with $\deg f(x) \geq 1$. Let $\overline{f}(x)$ be the polynomial in $\mathbb{Z}_p[x]$ obtained from $f(x)$ by reducing all the coefficients of $f(x)$ modulo p. If $\overline{f}(x)$ is irreducible over $\mathbb{Z}_p$ and $\deg \overline{f}(x) = \deg f(x)$, then $f(x)$ is irreducible over $\mathbb{Q}$.

(2) (a) Refer to the definition of irreducibility to determine why p must be prime.
(b) The implication is written in the form $(p \wedge q) \to r$. Prove this is logically equivalent to $(\neg r \wedge q) \to \neg p$ by remembering your young and idealistic days in Math 230.
(c) Given that -1 is a root of $f(x) = x^3 + x^2 + -5x + 3$ over $\mathbb{Q}$, reduce $\overline{f}(x)$ over $\mathbb{Z}_2$, and again over $\mathbb{Z}_3$ and $\mathbb{Z}_7$ ($\overline{f}(x)$ depends on $\mathbb{Z}_p$).
(d) What does the mod 2 irreducibility test tell you about $f(x) = x^5 + 2x + 4$?

Theorem 17.4: Eisenstein's Criterion.
Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0 \in \mathbb{Z}[x]$. If there is a prime p such that $p \nmid a_n$, $p \mid a_{n-1}, \ldots, p \mid a_0$, and $p^2 \nmid a_0$, then $f(x)$ is irreducible over $\mathbb{Q}$.

(3) What does Theorem 17.4 tell us about the following polynomials?
(a) $x^5 + 9X^4 + 12X^2 + 6$
(b) $x^4 + x + 1$
(c) $x^4 + 3x^2 + 9$

Corollary: Irreducibility of pth Cyclotomic Polynomial.
For any prime p, the pth cyclotomic polynomial

$$\Phi_p(x) = \frac{x^p - 1}{x - 1} = x^{p-1} + x^{p-2} + \cdots + x + 1$$

is irreducible over $\mathbb{Q}$.