Definition. Let D be an integral domain. Let $f(x) \in D[x]$ be a nonzero, nonunit polynomial.
(1) $f(x)$ is *irreducible* over D if writing $f(x) = g(x)h(x)$ with $g(x), h(x) \in D[x]$ implies that either $g(x)$ or $h(x)$ is a unit in $D[x]$.
(2) $f(x)$ is *reducible* if it is not irreducible.

(1) (a) Verify that $2x^2 + 4$ is irreducible over $\mathbb{R}$ but reducible over $\mathbb{C}$.
(b) Verify that $x^2 - 2$ is irreducible over $\mathbb{Q}$ but reducible over $\mathbb{R}$.

(2) Find a root of $x^2 + 1$ over $\mathbb{Z}_5$.

Theorem 17.1: Reducibility Test for Degrees 2 and 3.
Let F be a field. If $f(x) \in F[x]$ and $\deg f(x) = 2$ or 3, then $f(x)$ is reducible over F iff $f(x)$ has a zero in F.

(3) Why is the reverse implication easy?

(4) Use (2) and Theorem 17.1 to show that $x^2 + 1$ is reducible over $\mathbb{Z}_5$.

(5) Let p be prime. Describe a finite algorithm based on Theorem 17.1 for determining whether any degree 2 or 3 polynomial is reducible.

Definition. Let $f(x) = a_n x^n + \cdots a_1 x + a_0 \in \mathbb{Z}[x]$. The *content* of $f(x)$ is $\gcd(a_n, \ldots, a_0)$. We say $f(x)$ is *primitive* provided $f(x)$ has content 1.

(6) Which of the following polynomials are primitive: $x^5 + 9x^4 + 12x + 6$, $2x^2 + 4$, $2x^2 + 5x + 13$, $x^2 + \sqrt{2}$?

(7) **(a)** Multiply two primitive polynomials from (6). Is the result primitive?
(b) Multiply a primitive and a non-primitive polynomial from (6). Is the result primitive?

Gauss's Lemma.
The product of two primitive polynomials ____.

(8) Let $f(x) = x^5 + 9x^4 + 12x + 6$ and $g(x) = 2x^2 + 4$.
(a) Determine c_f, c_g, and c_{fg}, the contents of f, g, and fg, respectively.
(b) Find $\overline{f}(x) = f(x)/c_f$, and $\overline{g}(x) = g(x)/c_g$.
(c) Find $\overline{f \cdot g}(x) = f(x)g(x)/c_{fg}$ and show that this equals $\overline{f}(x)\overline{g}(x)$.