Corollary 1: The Remainder Theorem.
Let F be a field, $a \in F$, and $f(x) \in F[x]$. Then $f(a)$ is the remainder in the division of $f(x)$ by $x - a$.

Definition. Let F be a field, $a \in F$, and $f(x) \in F[x]$. We call a a *zero of multiplicity* k, where $k \geq 1$, provided $(x - a)^k$ is a factor of $f(x)$ but $(x - a)^{k+1}$ is not a factor of $f(x)$.

Corollary 2: The Factor Theorem.
Let F be a field, $a \in F$, and $f(x) \in F[x]$. Then a is a zero of $f(x)$ if and only if $x - a$ is a factor of $f(x)$.

(1) Prove Corollary 1 by writing $f(x) = (x-a)q(x)+r(x)$, determining what the Division Algorithm says about $r(x)$, and considering $f(a)$.

(2) Prove Corollary 2 directly from Corollary 1 and the definition that a is a zero of $f(x)$ provided $f(a) = 0$.

Corollary 3: Polynomials of Degree n Have at most n Zeros.
A polynomial of degree n over a field has at most n zeros counting multiplicity.

Theorem 16.3: $F[x]$ is a PID.
Let F be a field. Then $F[x]$ is a principal ideal domain.

Theorem 16.4: Criterion for $I = \langle g(x) \rangle$.
Let F be a field, I a nonzero ideal in $F[x]$, and $g(x)$ and element of $F[x]$. Then $I = \langle g(x) \rangle$ iff $g(x)$ is a nonzero polynomial of minimum degree in I.

(3) Let p be prime. Prove that in $\mathbb{Z}_p[x]$,

$$x^{p-1} - 1 = (x-1)(x-2)\cdots[x-(p-1)].$$

(4) Let $f(x) \in \mathbb{R}[x]$. If $f(a) = 0$ and $f'(a) = 0$ (where $f'(x)$ is the usual derivative of $f(x)$, and $f'(a)$ evaluates $f'(x)$ at $x = a$), show that $(x-a)^2$ divides $f(x)$ as follows:
(a) Divide $f(x)$ by $x - a$ using the division algorithm and apply the product rule of differentiation.
(b) Apply a theorem or corollary of the chapter to $f'(x)$, and bring the result back to $f(x)$.