Theorem 15.5: Homomorphism from $\mathbb{Z}$ to a Ring with Unity.
Let R be a ring with unity 1. Then the mapping $\phi : \mathbb{Z} \to R$ given by $n \to n{\cdot}1$ is a ring homomorphism.

Corollary 1: A Ring with Unity Contains $\mathbb{Z}_n$ or $\mathbb{Z}$.
If R is a ring with unity and the characteristic of R is $n > 0$, then R contains a subring isomorphic to $\mathbb{Z}_n$. If the characteristic of R is 0, then R contains a subring isomorphic to $\mathbb{Z}$.

Corollary 2: $\mathbb{Z}_n$ is a Homomorphic Image of $\mathbb{Z}$.
For any positive integer m the mapping of $\phi : \mathbb{Z} \to \mathbb{Z}_m$ given by $x \to x \mod m$ is a ring homomorphism.

Corollary 3: A Field contains $\mathbb{Z}_p$ or $\mathbb{Q}$.
If F is a field of characteristic p, then F contains a subfield isomorphic to $\mathbb{Z}_p$. If F is a field of characteristic 0, then F contains a subfield isomorphic to the rational numbers.

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(1) Given an integral domain D, the field of quotients of D is the set

$$S = \{(x, y) \,|\, x, y \in D,\ y \neq 0\},$$

with equivalence relation

$$(x_1, y_1) \equiv (x_2, y_2) \quad \text{iff} \quad x_1 y_2 = y_1 x_2,$$

addition and multiplication operations

$$\begin{aligned} (x_1, y_1) + (x_2, y_2) &= (x_1 y_2 + x_2 y_1, y_1 y_2) \\ (x_1, y_1)(x_2, y_2) &= (x_1 x_2, y_1 y_2). \end{aligned}$$

For the rest of the question, S is the field of quotients of the integral domain $D = \mathbb{Z}[i]$.
(a) Write down the set-builder notation for S, where $\mathbb{Z}[i] = \{a + bi \,|\, a, b \in \mathbb{Z}\}$.
(b) Write out the result of the multiplication and addition of two elements $(1 + 2i, 3 - i)$ and $(2 - 3i, 1 - 2i)$.
(c) For each of the two results from part (b), take the form $(a_1 + b_1 i, a_2 + b_2 i)$ and rewrite as $\dfrac{a_1 + b_1 i}{a_2 + b_2 i}$, rationalize the denominator, and express as $\alpha + \beta i$. In general, what set do α and β lie in?
(d) What other more commonly known field F is S isomorphic to, and how would you define the isomorphism $\phi : S \to F$?

(2) (Division algorithm in $F[x]$.) Recall that the division algorithm for dividing a polynomial $f(x)$ by a polynomial $g(x)$ is of the form

$$f(x) = g(x)q(x) + r(x),$$

where $g(x)$ is in the same polynomial ring, and either $r(x) = 0$ or the degree of $r(x)$ is less than that of $g(x)$. The resulting quotient $q(x)$ and remainder $r(x)$ are unique. This all holds true when F is a field.

Now define $f(x) = x^3 + 2x + 4$ and $g(x) = 3x + 2$.

(a) Find the quotient and remainder of dividing $f(x)$ by $g(x)$ when you view them as being elements of $\mathbb{R}[x]$.

(b) Now find the quotient and remainder of dividing $f(x)$ by $g(x)$ when you view them as being elements of $\mathbb{Z}_5[x]$. Carefully track the operations in the field used in part (a) to mimic the computation in $\mathbb{Z}_5[x]$.

(c) Find the quotient and remainder of dividing $f(x) = 5x^4 + 3x^3 + 1$ by $g(x) = 3x^2 + 2x + 1$ in $\mathbb{Z}_7[x]$.