Definition. A ring homomorphism $\phi$ from a ring $R$ to a ring $S$ is a mapping from $R$ to $S$ that preserves the two ring operations; that is, for all $a, b \in R$,

$$\phi(a + b) = \phi(a) + \phi(b) \quad \text{and} \quad \phi(ab) = \phi(a)\phi(b).$$

A ring homomorphism that is both one-to-one and onto is called a ring isomorphism.

(1) Let $\mathbb{R}[x]$ denote the ring of polynomials in $x$ with real coefficients. Prove that the mapping $f(x) \rightarrow f(1)$ is a ring homomorphism from $\mathbb{R}[x]$ to $\mathbb{R}$.

(2) Select a part of Theorem 15.1 to prove and write on the board. Refer to Theorems 10.1-2 for shortcuts.

(3) Let $\phi : 2\mathbb{Z} \rightarrow 3\mathbb{Z}$ be a mapping such that $\phi(2) = a$, where $a \in 3\mathbb{Z}$. Compare $\phi(2 + 2)$ to $\phi(2 \cdot 2)$ to show that $\phi$ cannot be a ring isomorphism.
Theorem 15.2: Kernels are Ideals.
Let \( \phi : R \rightarrow S \) be a ring homomorphism. Then \( \text{Ker} \phi = \{ r \in R | \phi(r) = 0 \} \) is an ideal of \( R \).

Theorem 15.3: First Isomorphism Theorem for Rings.
Let \( \phi : R \rightarrow S \) be a ring homomorphism. Then the mapping from \( R/\text{Ker} \phi \) to \( \phi(R) \), given by \( r + \text{Ker} \phi \rightarrow \phi(r) \), is an isomorphism. In symbols, \( R/\text{Ker} \phi \cong \phi(R) \).

Theorem 15.4: Ideals are Kernels.
Every ideal of a ring \( R \) is the kernel of a ring homomorphism of \( R \). In particular, an ideal \( A \) is the kernel of the mapping \( r \rightarrow r + A \) from \( R \) to \( R/A \).

Theorem 15.5: Homomorphism from \( \mathbb{Z} \) to a ring with unity.
Let \( R \) be a ring with unity \( 1 \). The mapping \( \phi(\mathbb{Z}) \rightarrow R \) given by \( n \rightarrow n \cdot 1 \) is a ring homomorphism.

(3) Let \( \phi : \mathbb{Z} \rightarrow M_2(\mathbb{Z}_3) \) be defined by \( \phi(n) = n \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \). What is \( \phi(\mathbb{Z}) \)?

(3) Let \( \phi : \mathbb{Z} \rightarrow M_2(\mathbb{R}) \) be defined by \( \phi(n) = n \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \). What is \( \phi(\mathbb{Z}) \)? What is the smallest set containing \( \phi(\mathbb{Z}) \) that is a ring that is closed under multiplicative inverses?

(4) Let \( \phi : M_2(\mathbb{Z}) \rightarrow \mathbb{Z} \) be defined by \( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \rightarrow a \). Prove or disprove that \( \phi \) is a ring homomorphism.