Definition. A *ring homomorphism* ϕ from a ring R to a ring S is a mapping from R to S that preserves the two ring operations; that is, for all $a, b \in R$,

$$\phi(a+b) = \phi(a) + \phi(b) \quad \text{and} \quad \phi(ab) = \phi(a)\phi(b).$$

A ring homomorphism that is both one-to-one and onto is called a *ring isomorphism*.

(1) Let $\mathbb{R}[x]$ denote the ring of polynomials in x with real coefficients. Prove that the mapping $f(x) \to f(1)$ is a ring homomorphism from $\mathbb{R}[x]$ to $\mathbb{R}$.

(2) Select a part of Theorem 15.1 to prove and write on the board. Refer to Theorems 10.1-2 for shortcuts.

(3) Let $\phi : 2\mathbb{Z} \to 3\mathbb{Z}$ be a mapping such that $\phi(2) = a$, where $a \in 3\mathbb{Z}$. Compare $\phi(2+2)$ to $\phi(2 \cdot 2)$ to show that ϕ cannot be a ring isomorphism.

Theorem 15.2: Kernels are Ideals.
Let $\phi : R \to S$ be a ring homomorphism. Then $\text{Ker}\phi = \{r \in R | \phi(r) = 0\}$ is an ideal of R.

Theorem 15.3: First Isomorphism Theorem for Rings.
Let $\phi : R \to S$ be a ring homomorphism. Then the mapping from $R/\text{Ker}\phi$ to $\phi(R)$, given by $r + \text{Ker}\phi \to \phi(r)$, is an isomorphism. In symbols, $R/\text{Ker}\phi \approx \phi(R)$.

Theorem 15.4: Ideals are Kernels.
Every ideal of a ring R is the kernel of a ring homomorphism of R. In particular, an ideal A is the kernel of the mapping $r \to r + A$ from R to R/A.

Theorem 15.5: Homomorphism from $\mathbb{Z}$ to a ring with unity.
Let R be a ring with unity 1. The mapping $\phi(\mathbb{Z}) \to R$ given by $n \to n \cdot 1$ is a ring homomorphism.

(3) Let $\phi : \mathbb{Z} \to M_2(\mathbb{Z}_3)$ be defined by $\phi(n) = n \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. What is $\phi(\mathbb{Z})$?

(3) Let $\phi : \mathbb{Z} \to M_2(\mathbb{R})$ be defined by $\phi(n) = n \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. What is $\phi(\mathbb{Z})$? What is the smallest set containing $\phi(\mathbb{Z})$ that is a ring that is closed under multiplicative inverses?

(4) Let $\phi : M_2(\mathbb{Z}) \to \mathbb{Z}$ be defined by $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \to a$. Prove or disprove that ϕ is a ring homomorphism.