(1) The ring $\mathbb{R}[x]$ is the set of polynomials in the variable x with real coefficients. The factor ring $\mathbb{R}[x]/\langle x^2+1\rangle$ is the ring

$$\{f(x)+\langle x^2+1\rangle | f(x)\in\mathbb{R}[x]\}.$$

(a) Use the division algorithm to prove that every coset in $\mathbb{R}[x]/\langle x^2+1\rangle$ has a representative of the form $ax+b$.
(b) Prove that every choice of $ax+b$ in (a) yields a distinct coset.
(c) By considering the cosets $x^2+\langle x^2+1\rangle$ and $-1+\langle x^2+1\rangle$, describe why we may say that x^2 "=" -1 in this factor ring. (This factor ring is a formulation of the complex numbers.)

Definition. A *prime ideal* A of a commutative ring R is proper ideal of R such that $a,b\in R$ and $ab\in A$ imply $a\in A$ or $b\in A$. A proper ideal A of a commutative ring R is called a *maximal ideal* of R if, whenever B is an ideal of R and $A\subseteq B\subseteq R$, then $B=A$ or $B=R$.

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(2) Let A be an ideal of the ring $\mathbb{Z}$. We wish to show that $A=\langle n\rangle$, for some $n\in\mathbb{Z}$, and then characterize the prime and maximal ideals.
(a) Recall that A is a group under addition. What does the Fundamental Theorem of Cyclic Groups (Thm. 4.3) say?
(b) Are the following prime and/or maximal: $\{0\},\mathbb{Z}$?
(c) Write the general form of a proper ideal of $\mathbb{Z}$ based on (a) and determine which are prime and which are maximal.

(3) Find all of the ideals in $\mathbb{Z}_{12}$. One way to do this is consider all ideals of the form $\langle a \rangle$, and then consider the smallest ideal containing b and $\langle a \rangle$ over all pairs a and b. (This trick won't verify all ideals are found for general rings.)

(4) In $\mathbb{Z} \oplus \mathbb{Z}$, let $I = \{(a, 0) | a \in \mathbb{Z}\}$.
(a) Show I is a prime ideal by considering $(a, 0)(b, 0) \in I$.
(b) Show that I is not maximal by augmenting I with an ideal in the second coordinate. (As a result we know that there exists a prime ideal that is not maximal. We will shortly see that for commutative rings with unity, all maximal ideals are prime.)

Theorem 14.3: R/A is an integral domain iff A is prime.
Let R be a commutative ring with unity and let A be an ideal of R. Then R/A is an integral domain iff A is prime.

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(5) Let A be an ideal of a ring R. Prove that $1 \in A$ implies $A = R$.

(6) Let R be a commutative ring with unity and let $a \in R$ be a unit. Why is a not a zero-divisor?

(7) Let R be a commutative ring, let A be an ideal of R, and let $b \in R \setminus A$. Prove that the set $B := \{br + a | b \in B, a \in A\}$ is an ideal of R properly containing A.

Theorem 14.4: R/A is an Field iff A is maximal.
Let R be a commutative ring with unity and let A be an ideal of R. Then R/A is a field iff A is maximal.

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(8) Briefly describe how we now know that in a commutative ring with unity, all maximal ideals are prime.