Definition. A subring A of a ring R is called a (two-sided) ideal of R if for every $r \in R$ and every $a \in A$ both ra and ar are in A.

Theorem 14.1: Ideal Test
A nonempty subset A of a ring R is an ideal of R if

1. $a - b \in A$ whenever $a, b \in A$.
2. ra and ar are in A whenever $a \in A$ and $r \in R$.

Definition. Let R be a commutative ring with unity. Then the set $\langle a \rangle = \{ra | r \in R\}$ is an ideal of R called the *principal ideal generated by* a.

(1) Let $\mathbb{R}[x]$ denote the set of all polynomials in x with real coefficients, and let A denote the subset of all polynomials with constant coefficient 0 (of the form $a_n x^n + \cdots + a_1 x^1 + 0$).
(a) Use the Ideal Test to show that A is an ideal of $\mathbb{R}[x]$.
(b) Use a subset equality proof to show that $A = \langle x \rangle$.

(2) Let R be a commutative ring with unity, and let $a_1, \ldots, a_n \in R$. Define $I = \langle a_1, \ldots, a_n \rangle := \{r_1 a_1 + \cdots r_n a_n | r_i \in R\}$.
(a) Show that I is an ideal of R.
(b) Show that if J is an ideal of R and $a_1, \ldots, a_n \in J$, then $I \subseteq J$. (Come back to this one if it is slowing you down.)

(3) Define $S = \{a + bi | a, b \in \mathbb{Z},\ b \text{ is even}\}$. Show that S is a subring but not an ideal of $\mathbb{Z}[i]$.

Defining Factor Rings. We want to define factor rings analogous to factor groups, so we can have some homomorphism theorems. Given a subring A of R and elements $s, t \in R$, we need the definitions of the following structures:

$$\begin{aligned}(s+A) &:= \{s+a | a \in A\}\\ (s+A)+(t+A) &:= (s+t)+A\\ (s+A)(t+A) &:= st+A.\end{aligned}$$

Theorem 14.2: Factor Rings
Let R be a ring and A a subring of R. The set of cosets $\{r+A | r \in R\}$ is a ring under the above operations iff A is an ideal in R.

(4) (a) What does commutativity of addition tell us about $\{r+A|r \in R\}$ under $+$?
(b) Show that if A is an ideal, then multiplication of cosets is well-defined; i.e., that $s+A = s'+A$ and $t+A = t'+A$ implies that $(s+A)(t+A) = (s'+A)(t'+A)$.
(c) If A is not an ideal, then there exists $r \in R$ such that either $ar \notin R$ or $ra \notin R$. Supposing that $ar \notin R$, show that the multiplication $(a+A)(r+A)$ is not well-defined, by considering that $(a+A) = (0+A)$.

(5) Construct the addition and multiplication tables of $\mathbb{Z}/5\mathbb{Z}$.

(6) In the ring of integers, find a positive integer a such that
(a) $\langle a \rangle = \langle 3 \rangle \langle 4 \rangle$,
(b) $\langle a \rangle = \langle 6 \rangle \langle 8 \rangle$,
(c) $\langle a \rangle = \langle m \rangle \langle m \rangle$.