Group Members: ____

(1) The Klein 4-ring $(R, +, \cdot)$ is one of the two smallest non-commutative rings, where $R = \{0, a, b, c\}$, and the operations are defined by

$+$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

$\cdot$	0	a	b	c
0	0	0	0	0
a	0	a	0	a
b	0	b	0	b
c	0	c	0	c

(a) Find all unities, units, zero-divisors, or idempotents.
(b) Find the left and right unities, where a left unity 1_{L} satisfies $1_{\mathrm{L}}x = x$ for all $x \in R$, and a right unity 1_{R} satisfies $x1_{\mathrm{R}} = x$ for all $x \in R$.
(c) Find all subrings.
(d) Tweak the multiplication table to get the other order 4 non-commutative ring, and repeat (a)-(c).

Definition. The *characteristic* of a ring R written char R, is the least positive integer n such that $nx = 0$ for all $x \in R$. If no such n exists, we say char $R = 0$.

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(2) Find the characteristic of the following rings.
(a) $\mathbb{Z}_n$, where $n \in \mathbb{Z}^+$.
(b) $\mathbb{Z}$.
(c) $\mathbb{Z}_2[i]$, from Group Activity 13A (11).
(d) The Klein 4-ring in (1).

Theorem 13.3: Characteristic of a Ring with Unity. Let R be a ring with unity 1. If 1 has infinite order under addition, then the characteristic of R is 0. If 1 has order n under addition, then the characteristic of R is n.

Theorem 13.4: Characteristic of an Integral Domain. The characteristic of an integral domain is 0 or prime.

(3) Let F be a field with unity 1 and $|F| = 2^n$.
(a) Is the additive order of 1 finite or infinite?
(b) What do Theorems 13.3 and 13.4 say about char$F$?
(c) From $|F|$ and the group structure of F, determine char$F$.
(d) What if the field F has $|F| = p^n$ for some prime p?

(4) Suppose that R is a commutative ring without zero-divisors. For $x, y \in R$ and $n \in \mathbb{Z}^+$, we have $(nx)y = x(ny)$.
(a) Suppose a nonzero $x \in R$ has finite additive order. Is it possible for a distinct nonzero $y \in R$ to have infinite additive order?
(b) Suppose nonzero $x, y \in R$ both have finite additive orders n and m, respectively. Is it possible for $n \neq m$?

(5) Show that any finite field F has order p^n, where p is a prime as follows.
(a) Suppose distinct primes p, q both divide $|F|$. What does Theorem 9.5 say?
(b) Combine (a) with the result of (4). Where is the contradiction?