Group Members: ____

Definition. A *zero-divisor* of a ring R is an element $a \in R \setminus 0$ such that $ab = 0$ for some nonzero $b \in R$.

Definition. An *integral domain* is a commutative ring with unity and no zero-divisors.

Try to answer (1)-(3) without referring to the text, but if you can't, see Table 13.2 on page 254.

(1) Give an example of a zero-divisor in $\mathbb{Z}_{12}$ by demonstrating the multiplication.

(2) Give an example of a ring that is commutative but does not have a unity.

(3) Give an example of a ring that is commutative, has a unity, but is not an integral domain. For this ring, find a zero-divisor and demonstrate the multiplication.

(4) Write down the multiplication table of $\mathbb{Z}_5$. Is $\mathbb{Z}_5$ an integral domain?

(6) Write out the multiplication table for $\mathbb{Z}_6$. Among the nonzero elements, identify the units and the zero-divisors.

(7) Show that every nonzero element of $\mathbb{Z}_n$ is a unit or a zero-divisor.

(8) Let R be a finite commutative ring with unity. Prove that every nonzero element of R is either a zero-divisor or a unit. (Hint: consider the set $S = \{s \cdot r | r \in R\}$.)

Theorem 13.1: Cancellation.
Let a, b, and c belong to an integral domain. If $a \neq 0$ and $ab = ac$, then $b = c$.

Definition. A *field* is a commutative ring with unity in which every nonzero element is a unit.

---

(9) By referring to (4), determine whether $\mathbb{Z}_5$ a field.

(10) Give an example of an integer domain that is not a field.

Theorem 13.2: Finite Integral Domains are Fields.
A finite integral domain is a field.

Corollary.
For every prime p, $\mathbb{Z}_p$ is a field.

Corollary.
$\mathbb{Z}_n$ is a field iff n is prime.

(11) Define $\mathbb{Z}_2[i]$ to be the set $\{a + bi | a, b \in \mathbb{Z}_2\}$ under addition and multiplication with coefficients reduced modulo 2. Here, $i = \sqrt{-1}$. (We may think of starting with all possible polynomials in x with coefficients in $\mathbb{Z}_2$, with x then replaced by i.) Write out the multiplication table and determine whether $\mathbb{Z}_2[i]$ is a field, an integral domain but not a field, or a commutative ring but not an integral domain.

Definition. The *characteristic* of a ring R written char R, is the least positive integer n such that $nx = 0$ for all $x \in R$. If no such n exists, we say char $R = 0$.

Theorem 13.3: Characteristic of a Ring with Unity. Let R be a ring with unity 1. If 1 has infinite order under addition, then the characteristic of R is 0. If 1 has order n under addition, then the characteristic of R is n.