PRINT Last name:________________ First name:________________

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Math 431 Exam 1, Spring 2009

[Automatic course 100 for a proof of Fermat's Last Theorem here. It should fit... .]

I. Examples, Counterexamples and short answer. (5 pts ea.) Do not give proofs, but clearly indicate your proposed example or counterexample, or short answer where appropriate.

1. Give an example of a group homomorphism $\phi : \mathbb{Z} \to \mathbb{Z}$ that is 1-1 but not onto.

2. Let $\phi : \mathbb{Z} \to \mathbb{Z}_6$ be a group homomorphism. What are the possible sizes of the image of the homomorphism? In other words, what could $|\phi(\mathbb{Z})|$ be?

3. Give an example of a group of order 81 that has no element of order 9.

4. What are the Abelian groups of order 40 up to isomorphism?

5. Give an example of the following, clearly labeling each with the appropriate letter:
   (a) A ring R with no units;
   (b) A ring R, and a unit $a \in R$, where R has no zero-divisors;
   (c) A ring R, a unit $a \in R$, and a zero-divisor $d \in R$;
   (d) A ring R, a unit $a \in R$, a zero-divisor $d \in R$, and a nonzero $r \in R$ that is neither a unit nor a zero-divisor.

6. Recall that an idempotent is an element x of a ring R such that $x^2 = x$. Give an example of a ring R and an idempotent $x \in R$ such that x is neither the unity or the zero (additive identity).

7. Give an example of a field that is neither $\mathbb{Q}$, $\mathbb{R}$, nor $\mathbb{C}$.

8. Give an example of a ring with:
   (a) Characteristic 0;
   (b) Characteristic 2;
   (c) Characteristic 4.

II. Constructions and Algorithms. (8,8,6,8 points resp.) Do not write proofs, but do give clear, concise answers, including steps to algorithms where applicable.

9. Express the group (with data given here) as an internal direct product of cyclic subgroups and as isomorphic to an external direct product of cyclic groups.

10. Let $\phi : D_4 \to \mathbb{Z}_6$ be a group homomorphism from the dihedral group D_4 to the integers mod 6 defined by $\phi(R_{90}) = 0$ and $\phi(H) = 3$. Draw the homomorphism diagram/figure for ϕ as given in class (one circle for domain, one circle for range, arrows and labels, etc.) being sure to:
(a) Label the domain and image, and all elements thereof;
(b) Draw the partition of the domain into cosets of the kernel;
(c) Draw an arrow (or arrows) indicating the image of each element under ϕ; and
(d) Draw the partition of the range into the image $\phi(D_4)$ and its possibly nonempty complement.

11. Construct the multiplication table for the ring $R = \{0, 3, 6, 9, 12\}$ under addition and multiplication modulo 15. Clearly identify the unity (if present) and each unit along with its multiplicative inverse (if present).

12. The lattice of ideals of a ring R is drawn like a subgroup diagram of a group. Every ideal I of R is drawn as a point in the lattice, and two ideals I and J of R are connected by a line provided $I \subset J$ and there is no ideal K with $I \subset K \subset J$.
(a) Draw the lattice of ideals for $\mathbb{Z}_{18}$.
(b) Circle the ideals that are *prime*.
(c) Put a square around the ideals that are *maximal*.

III. Proofs. (10 pts ea.) Part of the score, including partial credit, is determined by careful formatting of the proof (forward and reverse directions, assumptions, conclusions, stating whether you are proving by direct proof, contrapositive, contradiction, induction, etc.).

13. Let $\mathbb{C}^*$ be the group of nonzero complex numbers under multiplication, and let $\mathbb{R}^+$ be the positive reals under multiplication. Define the complex unit circle $U := \{x \in \mathbb{C} \mid |x| = 1\}$. Prove that $\mathbb{C}^*/\mathbb{R}^+ \approx U$. (This is a group theory question. You may assume that $|xy| = |x||y|$, where $|a + bi| = \sqrt{a^2 + b^2}$. Name or describe any theorems from the book you wish to use without proof.)

14. Consider $\mathbb{Q}$ as a ring under the usual addition and multiplication. Prove or disprove that $S = \{m/n \in \mathbb{Q} \,|\, m, n \in \mathbb{Z}, \text{and } n \text{ odd}\}$ is a subring of $\mathbb{Q}$.

15. Let R be a commutative ring, let $a \in R$ be a zero-divisor of R, and let n be a positive integer. Prove that a^n is either zero or a zero-divisor.