Theorem 8.2 (Criterion for $G \oplus H$ to be cyclic)

Let $G$ and $H$ be finite cyclic groups. Then $G \oplus H$ is cyclic iff $|G|$ and $|H|$ are relatively prime.

Proof ($\Rightarrow$)

Assume $G \oplus H$ is cyclic. Let $|G| = m$ and $|H| = n$.

Let $(g, h) \in G \oplus H$ with $\langle (g,h) \rangle = G \oplus H$.

Then $|(g,h)| = |G \oplus H| = |G| \cdot |H| = m \cdot n$.

Now write $d = \gcd(m, n)$.

Compute $(g, h)^{mn/d} = \left( (g^m)^{n/d}, (h^n)^{m/d} \right)$

$$= (e_G, e_H)$$

To get $|(g,h)| \leq mn/d$

$$d \leq \frac{mn}{|(g,h)|} = 1$$

so that $\gcd(m,n) = 1$.

($\Leftarrow$) Since $G, H$ cyclic, let $G = \langle g \rangle$, $H = \langle h \rangle$.

Assume $\gcd(|G|, |H|) = \gcd(m,n) = 1$.

$\gcd(m,n) \operatorname{lcm}(m,n) = m \cdot n \Rightarrow \operatorname{lcm}(m,n) = m \cdot n$.

By Theorem 8.1, $|(g,h)| = \operatorname{lcm}(|g|, |h|) = m \cdot n$

$= |G \oplus H|$, and so

$G \oplus H = \langle (g,h) \rangle$. $\square$

<u>Theorem 8.1</u> $G_1, G_2, \ldots, G_n$ finite groups, and $g_1 \in G_1, \ldots, g_n \in G_n \Rightarrow$

$$|(g_1, g_2, \ldots, g_n)| = \operatorname{lcm}(|g_1|, \ldots, |g_n|).$$

(component wise group operation in $G_1 \oplus G_2 \oplus \cdots \oplus G_n$.)

<u>Proof</u>

Recall Cor 2 of Thm 4.1 (p.75): $a^k = e \Rightarrow |a| \mid k$.

The identity of $G_1 \oplus \cdots \oplus G_n$ is $(e_1, \ldots, e_n)$, where $e_i$ is the identity of $G_i$ for all $i$ (Exercise).

Now Suppose $t \in \mathbb{Z}^+$ is such that

$$(g_1, \ldots, g_n)^t = (g_1^t, \ldots, g_n^t) = (e_1, \ldots, e_n).$$

For fixed $i$, $g_i^t = e_i \Rightarrow |g_i| \mid t$ (Cor 2 of Thm 1)

Therefore $t$ is a common multiple of

$$|g_1|, |g_2|, \ldots, |g_n|.$$

The smallest positive such multiple is by definition

$$l = \operatorname{lcm}(|g_1|, |g_2|, \ldots, |g_n|).$$

Indeed, $(g_1, \ldots, g_n)^l = (g_1^l, \ldots, g_n^l) = (e_1, \ldots, e_n)$ since $|g_i| \mid l \Rightarrow g_i^l = e_i$ for all $i$. $\square$