Theorem 0.2. GCD Is a Linear Combination (integer linear combination).
For any nonzero integers a and b, there exist integers s and t such that

$$\gcd(a, b) = as + bt\,.$$

Moreover, $\gcd(a, b)$ is the smallest positive integer of the form $as + bt$.

Euclidean Algorithm. If a, b are positive integers, we may find $\gcd(a, b)$ by repeated use of the division algorithm to produce a decreasing sequence of integers $r_1 > r_2 > \cdots > r_k > 0$, where the last nonzero remainder r_k is equal to $\gcd(a, b)$.

$$\begin{aligned}
a &= bq_1 + r_1 && 0 < r_1 < b \\
b &= r_1q_2 + r_2 && 0 < r_2 < r_1 \\
r_1 &= r_2q_3 + r_3 && 0 < r_3 < r_2 \\
&\vdots && \vdots \\
r_{k-3} &= r_{k-2}q_{k-1} + r_{k-1} && 0 < r_{k-1} < r_{k-2} \\
r_{k-2} &= r_{k-1}q_{k-1} + r_k && 0 < r_k < r_{k-1} \\
r_{k-1} &= r_kq_{k+1} + 0
\end{aligned}$$

Theorem 0.2. Euclid's Lemma.
If p is a prime that divides ab, then p divides a or p divides b.

Theorem 0.3. Fundamental Theorem of Arithmetic.
Every integer greater than 1 is a prime or a product of primes. This product is unique, except for the order in which the factors appear. Thus if

$$\begin{aligned}
n &= p_1p_2\cdots p_r\,, \qquad \text{and} \\
n &= q_1q_2\cdots q_s\,,
\end{aligned}$$

where the p's and q's are primes, then $r = s$ and, after renumbering the q's, we have $p_i = q_i$ for all i.

Proof of (9). Let S be the set of positive integers n satisfying

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

Base case.
S contains 1 since $1 = \frac{1(1+1)}{2}$.
Inductive step.
Let n be a positive integer and suppose that $n \in S$.
By this assumption,

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

Adding $n+1$ to both sides, we have

$$\begin{aligned} 1 + 2 + \cdots + n + (n+1) &= \frac{n(n+1)}{2} + (n+1) \\ 1 + 2 + \cdots + n + (n+1) &= \frac{n^2 + n + 2n + 1}{2} \\ 1 + 2 + \cdots + n + (n+1) &= \frac{n^2 + 3n + 1}{2} \\ 1 + 2 + \cdots + n + (n+1) &= \frac{(n+1)[(n+1)+1]}{2}. \end{aligned}$$

Therefore $n + 1 \in S$.

By the First Principle of Mathematical Induction, S contains all positive integers. □

Proof of (10). By exhaustion, we analyze the small postage values to see which can be composed of 4 and 9 cent stamps.

1			15	
2			16	$4 \cdot 4$
3			17	$1 \cdot 9 + 2 \cdot 4$
4	$1 \cdot 4$		18	$2 \cdot 9$
5			19	
6			20	$5 \cdot 4$
7			21	$1 \cdot 9 + 3 \cdot 4$
8	$2 \cdot 4$		22	$2 \cdot 9 + 1 \cdot 4$
9	$1 \cdot 9$		23	
10			24	$6 \cdot 4$
11			25	$1 \cdot 9 + 4 \cdot 4$
12	$3 \cdot 4$		26	$2 \cdot 9 + 2 \cdot 4$
13	$1 \cdot 9 + 1 \cdot 4$		27	$3 \cdot 9$
14			28	$7 \cdot 4$

Sketch of the rest of proof. Argue that $24, 25, 26, 27 \in S$, and for $n > 27$, $n - 4 \in S \Rightarrow n \in S$. The Second Principle of Mathematical Induction gives the desired result, namely, that 23 is the largest amount that cannot be composed of 4 and 9 cent stamps.

Question. How is the question related to expressing

$$\begin{aligned} 1 &= 4 \cdot s + 9 \cdot t, \\ -1 &= 4 \cdot s' + 9 \cdot t' \quad ? \end{aligned}$$