

Theorem 7.3 Orbit-Stabilizer Theorem

Let G be a finite group of permutations of a set S . Then for any $i \in S$,

$$|G| = |\text{orb}_G(i)| \cdot |\text{stab}_G(i)|.$$

Proof Let

$\ell_1 \text{stab}_G(i), \ell_2 \text{stab}_G(i), \dots, \ell_r \text{stab}_G(i)$
 be the distinct left cosets of $\text{stab}_G(i)$ in G .

By Lagrange's Theorem, $r = |G : \text{stab}_G(i)| = |G| / |\text{stab}_G(i)|$.

We construct and prove a bijection

$$T : \{\ell_j \text{stab}_G(i) \mid j = \{1, \dots, r\}\} \rightarrow \text{orb}_G(i).$$

$$\text{Define } T(\ell \text{stab}_G(i)) = \ell(i).$$

Claim 1 T is well-defined.

(We need this because there are many choices of ℓ to produce the same left coset $\ell \text{stab}_G(i)$).

Suppose $\ell_x \text{stab}_G(i) = \ell_y \text{stab}_G(i)$.

By Lemma p.139 part 5, $\ell_x^{-1} \ell_y \in \text{stab}_G(i)$.

By definition, this means $\ell_x^{-1} \ell_y(i) = i$

$$\ell_x^{-1} \ell_y(i) = \ell_x^{-1}(\ell_y(i)) = i$$

iff $\ell_x(i) = \ell_y(i)$ by defn. inverse.

Therefore $\ell_x \text{stab}_G(i) = \ell_y \text{stab}_G(i) \Rightarrow$

$$T(\ell_x \text{stab}_G(i)) = \ell_x(i) = \ell_y(i) = T(\ell_y \text{stab}_G(i))$$

and T is well-defined.

Claim 2 T is 1-1.

Let $\varphi_x \text{stab}_G(i), \varphi_y \text{stab}_G(i)$ be two left cosets.

Assume $T(\varphi_x \text{stab}_G(i)) = T(\varphi_y \text{stab}_G(i))$.

By definition of T , $\varphi_x(i) = \varphi_y(i)$.

φ_x is a permutation, and so has inverse φ_x^{-1} .

By definition of inverse function,

$$\varphi_x(i) = \varphi_y(i) \text{ iff } \varphi_x^{-1}(\varphi_y(i)) = i,$$

$$\varphi_x^{-1}\varphi_y(i) = \varphi_x^{-1}(\varphi_y(i)) = i \Rightarrow \varphi_x^{-1}\varphi_y \in \text{stab}_G(i).$$

By lemma p.139 part 5, $\varphi_x \text{stab}_G(i) = \varphi_y \text{stab}_G(i)$

and T is 1-1.

Claim 3 T is onto.

Let $j \in \text{orb}_G(i)$.

Then there exists some $\varphi \in G$ with $\varphi(i) = j$.

Simply notice $T(\varphi \text{stab}_G(i)) = \varphi(i) = j$. \square

Rotations of a tetrahedron

Label the sides 1, 2, 3, 4. $S = \{1, 2, 3, 4\}$

How many permutations stabilize 1?

$\epsilon, (234), (243)$

What is the orbit of 1?

$\{1, 2, 3, 4\}$

$$|\text{stab}_G(1)| \cdot |\text{orb}_G(1)| = 3 \cdot 4 = 12 = |A_4| \checkmark$$

Corollary There are — rotations of the cube.

Chapter 7 (3)
Math 430
Lagrange's Thm

The set of faces of the cube is $S = \{1, 2, 3, 4, 5, 6\}$
(we can think of these as the points in the center of the respective faces.)

Set $i = 1$. Let G be the rotations of the cube.
 $\text{Stab}_G(i) = 4$ since the axis through i and the opposite point has $0^\circ, 90^\circ, 180^\circ$, and 270° rotations which stabilize i .

Clearly $\text{orb}_G(i) = \{1, 2, 3, 4, 5, 6\}$.

Therefore $|G| = 4 \cdot 6 = 24$.

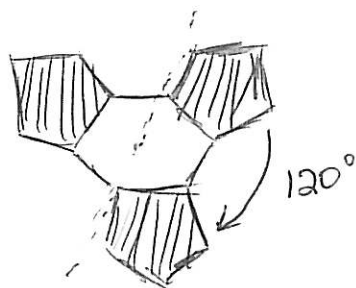
Corollary There are — rotations of the truncated icosahedron (soccer ball)

There are 20 hexagons and 12 pentagons on the truncated icosahedron. Let $G = \text{rotations}$.

Let $i \in S = \{ \text{hexagons} \}$

$j \in T = \{ \text{pentagons} \}$

	$\text{stab}_G(i)$	$\text{orb}_G(i)$	product
hex. i	3 (??)	20	60
pent. j	5	12	60



The 60° rotation does not send pentagons to pentagons.