Theorem 6.1 Cayley's Theorem

Every group is isomorphic to a group of permutations.

Proof

Let G be any group.

We need to construct a group $\overline{G}$ of permutations, with an isomorphism $\varphi: G \to \overline{G}$.

Q. Permutations on what? A. on G.

Idea $\quad g \longleftrightarrow T_g$

element of G $\quad$ permutation on G

Definition For each $g \in G$, define

$$T_g : G \to G \quad \text{by}$$

$$T_g(x) = gx \quad \text{for all } x \in G.$$

Fact 1 T_g is a permutation on G (Ex. 21)

Fact 2 Set $\overline{G} = \{ T_g \mid g \in G \}$.

Then $\overline{G}$ is a group under function composition.

Fact 2 Proof

We claim $T_g T_h = T_{gh}$, so that function composition is a closed binary operation.

$$T_g T_h(x) = T_g(hx) = ghx$$
$$= (gh)x = T_{gh}(x).$$

$gh \in G \Rightarrow T_{gh} \in \overline{G}$, so it's closed.

associativity Yes, b/c function composition.

identity $T_e T_g(x) = T_{eg}(x) = T_{ge}(x) = T_g(x)$
$= T_g T_e(x)$; so
$T_e \in \overline{G}$ is the identity in $\overline{G}$.

inverses $T_g T_{g^{-1}}(x) = T_{gg^{-1}}(x) = T_e(x)$
$= T_{g^{-1}g}(x) = T_{g^{-1}} T_g(x)$,
so $(T_g)^{-1} = T_{g^{-1}}$ is the inverse of T_g.

Isomorphism Define $\varphi: G \to \overline{G}$
by $\varphi(g) = T_g$.

Fact 3 φ is a bijection (exercise)

Fact 4 φ is operation preserving
$\varphi(xy) = T_{xy} = T_x T_y = \varphi(x)\varphi(y)$. $\square$